is still equal to one, see equation (10.131). Consequently, the absolute value of the first root of equation (10.131) will be greater than one, while that of the second root will be less than one. Let us denote $|q_1(\lambda)| < 1$ and $|q_2(\lambda)| > 1$. The general solution of equation (10.130) has the form:

$$u_m = c_1 q_1^m + c_2 q_2^m,$$

where c_1 and c_2 are arbitrary constants. Accordingly, the general solution that satisfies additional constraint (10.126), i.e., that decays as $m \longrightarrow +\infty$, is written as

$$u_m = c_1 q_1^m, \quad |q_1| = |q_1(\lambda)| < 1,$$

and the general solution that satisfies additional constraint (10.128), i.e., that decays as $m \longrightarrow -\infty$, is given by

$$u_m = c_2 q_2^m, \quad |q_2| = |q_2(\lambda)| > 1.$$

To calculate the eigenvalues of problem (10.125), one needs to substitute $u_m^p = c_1 \lambda^p q_1^m$ into the left boundary condition $\boldsymbol{l}_1 u_0 = 0$ and find those q_1 and λ, for which it is satisfied. If, for example, $\boldsymbol{l}_1 u_0 \equiv u_0 = 0$, then $c_1 \lambda^p q_1^0 = 0$ implies $\lambda = 0$, because $c_1 = 0$ would mean a zero eigenfunction. Thus, $\lambda = 0$ is an eigenvalue provided that $r < 1/4$, because only in this case for $\lambda = 0$ we may have $|q_1| < 1$. Otherwise, problem (10.125) has no eigenvalues. Likewise, if $\boldsymbol{l}_1 u_0 \equiv u_1 - u_0 = 0$, then $c_1 \lambda^p (q_1 - q_1^0) = c_1 \lambda^p (q_1 - 1) = 0$ yields either $\lambda = 0$ for $r < 1/4$ or otherwise no eigenvalues because $c_1 \neq 0$ and $q_1 \neq 1$. If, however, $\boldsymbol{l}_1 u_0 \equiv 2u_1 - u_0 = 0$, then condition $c_1 \lambda^p (2q_1 - q_1^0) = c_1 \lambda^p (2q_1 - 1) = 0$ is satisfied for $c_1 \neq 0$ and $q_1 = 1/2 < 1$. Substituting $q_1 = 1/2$ into the characteristic equation (10.131) we find that

$$\lambda = 1 + r\left(q_1 - 2 + \frac{1}{q_1}\right) = 1 + \frac{r}{2}.$$

This is the only eigenvalue of problem (10.125). It does not belong to the unit disk on the complex plane, and therefore the necessary stability condition is violated.

The eigenvalues of the auxiliary problem (10.127) are calculated analogously. They are found from the equation $\boldsymbol{l}_2 u_M = 0$ when

$$u_m = c_2 q_2^m, \quad |q_2| = |q_2(\lambda)| > 1, \quad m = M, M-1, M-2, \ldots.$$

For stability, it is necessary that they all belong to the unit disk on the complex plane.

We can now provide more specific comments following Remark 10.1. When boundary condition $\boldsymbol{l}_1 u_0 \equiv 2u_1 - u_0 = 0$ is employed in problem (10.125) then the solution that satisfies condition (10.126) is found in the form $u_m^p = \lambda^p q_1^m$, where $q_1 = 1/2$ and $\lambda = 1 + r/2 > 1$. This solution is only defined for $m \geq 0$. If, however, we were to extend it to the region $m < 0$, we would have obtained an unbounded function: $u_m^p \longrightarrow \infty$ as $m \longrightarrow -\infty$. In other words, the function $u_m^p = \lambda^p q_1^m$ cannot be used in the framework of the standard von Neumann analysis of problem (10.109).

This consideration leads to a very simple explanation of the mechanism of instability. The introduction of a boundary condition merely expands the pool of candidate