The following theorem due to Kreiss provides a sufficient condition of stability for dissipative finite-difference schemes (see [RM67, Chapter 5] for the proof):

***THEOREM 10.5 (Kreiss)***

*Let the matrix $\boldsymbol{A}(x)$ in equation (10.120) and the matrices $\boldsymbol{A}_j(x,h)$, $j = -j_{\text{left}}, \dots, j_{\text{right}}$, in equation (10.121) be Hermitian, uniformly bounded, and uniformly Lipshitz-continuous with respect to $x$. Then, if scheme (10.121) is dissipative of order $2d$ in the sense of Definition 10.5, and has accuracy of order $2d-1$ for some positive integer $d$, then this scheme is stable in $l_2$.*

Having formulated Theorem 10.5, we can revisit our previously analyzed examples. The first order upwind scheme (10.115) of Example 1 is dissipative of order $2d = 2$ when $r = \tau/h < 1$. Its accuracy is $\mathscr{O}(h)$, and therefore we can set $d = 1$, apply Theorem 10.5, and conclude that this scheme is stable. The same conclusion can obviously be made regarding the implicit scheme (10.119) analyzed in Example 4. Moreover, in this case the dissipation of the scheme puts no constraints on the value of $r$, and therefore the stability is unconditional.

The Lax-Wendroff scheme (10.117) of Example 2 is dissipative of order $2d = 4$ when $r < 1$, i.e., we need to consider $d = 2$. However, this scheme is only second order accurate, while $2d - 1 = 3$. Therefore, Theorem 10.5 does not allow us to immediately judge the stability of scheme (10.117).[6]

The Crank-Nicolson scheme (10.118) of Example 3 is non-dissipative, and therefore Theorem 10.5 does not apply.

On the other hand, let us note that according to Theorem 10.3 (proven in Section 10.3.5), all four schemes discussed in Examples 1–4 are stable in the sense of $l_2$ when they satisfy the von Neumann condition. Recall, Theorem 10.3 says that a scalar constant-coefficient finite-difference scheme is stable in $l_2$ if and only if the von Neumann condition holds. Consequently, the explicit upwind scheme and the Lax-Wendroff scheme are stable provided that $r \leq 1$, while the Crank-Nicolson scheme and the implicit upwind scheme are stable for all values of $r$.

As such, we see that for the case of scalar constant-coefficient finite-difference equations, the result of Theorem 10.5 may, in fact, be superseded by that of Theorem 10.3. We know, however, that for systems the von Neumann criterion alone provides only a necessary condition for stability, see Section 10.3.6. Moreover, for the case of variable coefficients, the principle of frozen coefficients, see Section 10.4.1, also provides a necessary condition for stability only. These are the cases when the sufficient condition given by Theorem 10.5 is most helpful.

Note also that there is a special large group of methods used in particular for the computation of fluid flows, when dissipation (viscosity) is artificially added to the scheme to improve its stability characteristics, see Section 11.2.1.

[6] The Theorem can still be applied though, but only after a change of variables, see [RM67, Chapter 5].