Unlike the schemes discussed in Examples 1 & 2, the Crank-Nicolson scheme (10.118) is implicit. It approximates problem (10.116) with second order accuracy with respect to h, provided that $r = \tau/h = \text{const}$.

For the spectrum of scheme (10.118) we can easily find:

$$\frac{\lambda - 1}{\tau} - \frac{1}{2}\left[\lambda \frac{e^{i\alpha} - e^{-i\alpha}}{2h} + \frac{e^{i\alpha} - e^{-i\alpha}}{2h}\right] = 0,$$

which yields:

$$\lambda(\alpha) = \left(1 + \frac{ir}{2}\sin\alpha\right) \cdot \left(1 - \frac{ir}{2}\sin\alpha\right)^{-1}.$$

Consequently,

$$|\lambda(\alpha)| = 1$$

irrespective of the specific value of r. Therefore, the Crank-Nicolson scheme (10.118) is non-dissipative. We also see that it satisfies the von Neumann stability condition (10.78).

Example 4

Finally, consider a fully implicit first order upwind scheme (10.105):

$$\begin{gathered} \frac{u_m^{p+1} - u_m^p}{\tau} - \frac{u_{m+1}^{p+1} - u_m^{p+1}}{h} = 0, \\ u_m^0 = \psi(x_m), \quad m = 0, \pm 1, \pm 2, \dots, \quad p = 0, 1, 2, \dots, [T/\tau] - 1, \end{gathered} \tag{10.119}$$

for the same Cauchy problem (10.116). Substituting the solution in the form $u_m^p = \lambda^p e^{i\alpha m}$, $-\pi \le \alpha \le \pi$, into the difference equation of (10.119), we obtain:

$$\frac{\lambda - 1}{\tau} - \lambda \frac{e^{i\alpha} - 1}{h} = 0,$$

which immediately yields the spectrum of the scheme (10.119):

$$\lambda(\alpha) = \frac{1}{1 + r - re^{i\alpha}}, \quad r = \tau/h = \text{const}.$$

Using the inequality: $|\alpha|/4 \le |\sin(\alpha/2)|$, $\alpha \in [-\pi, \pi]$, from Example 1, we have:

$$\begin{aligned} |\lambda|^2 &= \frac{1}{(1 + r - r\cos\alpha)^2 + r^2\sin^2\alpha} = \frac{1}{1 + 4r(1+r)\sin^2\frac{\alpha}{2}} \\ &= 1 - \frac{4r(1+r)\sin^2\frac{\alpha}{2}}{1 + 4r(1+r)\sin^2\frac{\alpha}{2}} \le 1 - \frac{4r(1+r)\sin^2\frac{\alpha}{2}}{1 + 4r(1+r)} \le 1 - \frac{4r(1+r)}{1 + 4r(1+r)}\frac{|\alpha|^2}{16}. \end{aligned}$$

Consequently, if we introduce $\delta > 0$ according to:

$$2\delta = \frac{1}{16} \cdot \frac{4r(1+r)}{1 + 4r(1+r)},$$