Using the previous two expressions, we differentiate the function $\varphi(t)$ defined by formula (2.24) $s+1$ times and obtain:

$$\varphi^{(s+1)}(t) = f^{(s+1)}(t) - k(s+1)!.$$

Substituting $t = \xi$ into the last equality, and recalling that $\varphi^{(s+1)}(\xi) = 0$, we arrive at the following expression for k:

$$k = \frac{f^{(s+1)}(\xi)}{(s+1)!}.$$

Finally, by substituting k into equality (2.26) we obtain a formula for $R_s(\bar{t})$ that would actually coincide with formula (2.23) because $\bar{t} \in [\alpha, \beta]$ has been chosen arbitrarily. ∎

THEOREM 2.6

Under the assumptions of the previous theorem, the following estimate holds:

$$\max_{\alpha \le t \le \beta} |R_s(t)| \le \frac{1}{(s+1)!} \max_{\alpha \le t \le \beta} |f^{(s+1)}(t)| (\beta - \alpha)^{s+1}. \tag{2.27}$$

PROOF We first note that $\forall t \in [\alpha, \beta]$ the absolute value of each expression $t - t_0$, $t - t_1$, ..., $t - t_s$ will not exceed $\beta - \alpha$. Then, we use formula (2.23):

$$\begin{aligned} |R_s(t)| &= \frac{1}{(s+1)!} |f^{(s+1)}(\xi)(t - t_0)(t - t_1) \dots (t - t_s)| \\ &\le \frac{1}{(s+1)!} \max_{\alpha \le t \le \beta} |f^{(s+1)}(t)| (\beta - \alpha)^{s+1}. \end{aligned} \tag{2.28}$$

As $t \in [\alpha, \beta]$ on the left-hand side of formula (2.28) is arbitrary, the required estimate (2.27) follows. ∎

Let us emphasize that we have proven inequality (2.27) for an arbitrary distribution of the (distinct) interpolation nodes $t_0, t_1, \dots, t_s$ on the interval $[\alpha, \beta]$. For a given fixed distribution of nodes, estimate (2.27) can often be improved. For example, consider a piecewise linear interpolation and assume that the nodes t_0 and t_1 coincide with the endpoints α and β, respectively, of the interval $\alpha \le t \le \beta$. Then,

$$\begin{aligned} |R_1(t)| &= \left| \frac{f''(\xi)}{(s+1)!} (t - \alpha)(t - \beta) \right| \\ &\le \frac{1}{2} \max_{\alpha \le t \le \beta} |f''(t)| \max_{\alpha \le t \le \beta} |(t - \alpha)(t - \beta)| = \frac{1}{8} \max_{\alpha \le t \le \beta} |f''(t)| (\beta - \alpha)^2, \end{aligned}$$

which yields

$$\max_{\alpha \le t \le \beta} |R_1(t)| \le \frac{1}{8} \max_{\alpha \le t \le \beta} |f''(t)| (\beta - \alpha)^2, \tag{2.29}$$