

Equivalently, we can require that

$$|\lambda(\alpha)| \leq 1 + c_1 \tau, \quad (10.78)$$

where  $c_1$  is a constant that does not depend either on  $\alpha$  or on  $\tau$ . Inequality (10.78) represents the necessary spectral condition for stability due to von Neumann. It is called spectral because of the following reason.

Existence of the solution in the form (10.76) shows that the harmonic  $e^{i\alpha m}$  is an eigenfunction of the transition operator from time level  $t_p$  to time level  $t_{p+1}$ :

$$u_m^{p+1} = (1-r)u_m^p + ru_{m+1}^p, \quad m = 0, \pm 1, \pm 2, \dots$$

According to the finite-difference equation (10.73), this operator maps the grid function  $\{u_m^p\}$ ,  $m = 0, \pm 1, \pm 2, \dots$ , defined for  $t = t_p$  onto the grid function  $\{u_m^{p+1}\}$ ,  $m = 0, \pm 1, \pm 2, \dots$ , defined for  $t = t_{p+1}$ . The quantity  $\lambda(\alpha)$  given by formula (10.77) is therefore an eigenvalue of the transition operator that corresponds to the eigenfunction  $\{e^{i\alpha m}\}$ . In the literature,  $\lambda(\alpha)$  is sometimes also referred to as the amplification factor, we have encountered this concept in Section 10.2.3. The set of all complex numbers  $\lambda = \lambda(\alpha)$  obtained when the parameter  $\alpha$  sweeps through the real axis forms a curve on the complex plane. This curve is called the spectrum of the transition operator.

Consequently, the necessary stability condition (10.78) can be re-formulated as follows: The spectrum of the transition operator that corresponds to the difference equation of problem (10.73) must belong to the disk of radius  $1 + c_1 \tau$  centered at the origin on the complex plane. In our particular example, the spectrum (10.77) does not depend on  $\tau$  at all. Therefore, condition (10.78) is equivalent to the requirement that the spectrum  $\lambda = \lambda(\alpha)$  belong to the unit disk:

$$|\lambda(\alpha)| \leq 1. \quad (10.79)$$

Let us now use the criterion that we have formulated, and actually analyze stability of problem (10.69)–(10.70). The spectrum (10.77) forms a circle of radius  $r$  centered at the point  $(1-r, 0)$  on the complex plane. When  $r < 1$ , this circle lies inside the unit disk, being tangent to the unit circle at the point  $\lambda = 1$ . When  $r = 1$  the spectrum coincides with the unit circle. Lastly, when  $r > 1$  the spectrum lies outside the unit disk, except one point  $\lambda = 1$ , see Figure 10.5. Accordingly, the necessary stability condition (10.79) is satisfied for  $r \leq 1$  and is violated for  $r > 1$ . Let us now recall that in Section 10.1.3 we have studied the same difference problem and have proven that it is stable when  $r \leq 1$  and is unstable when  $r > 1$ . Therefore, in this particular case the von Neumann necessary stability condition appears sufficiently sensitive to distinguish between the actual stability and instability.

In the case of general Cauchy problems for finite-difference equations and systems, we give the following

**DEFINITION 10.3** *The spectrum of a finite-difference problem is given by the set of all those and only those  $\lambda = \lambda(\alpha, h)$ , for which the corresponding*