leads to the previously analyzed upwind scheme (10.10). Another solution

$$a^0 = \frac{1}{rh} = \frac{1}{\tau}, \quad a_{-1} = \frac{1}{2h}, \quad a_0 = \frac{-1}{rh} = \frac{1}{h} - \frac{1}{\tau}, \quad a_1 = -\frac{1}{2h}$$

yields the scheme $\boldsymbol{L}_h u^{(h)} = f^{(h)}$ with the operator and the right-hand side defined as:

$$\boldsymbol{L}_h u^{(h)} = \begin{cases} \dfrac{u_m^{p+1} - u_m^p}{\tau} - \dfrac{u_{m+1}^p - u_{m-1}^p}{2h}, \\ u_m^0, \end{cases} \quad \text{and} \quad f^{(h)} = \begin{cases} \varphi(x_m, t_p), \\ \psi(x_m). \end{cases} \tag{10.51}$$

Given any solution of system (10.50), one, of course, needs to substitute it into the remainder of formula (10.49) and make sure that it is indeed small. For the previous two solutions that lead to the schemes (10.10) and (10.51), respectively, this substitution yields the remainder:

$$\frac{a^0 r^2 h^2}{2} \frac{\partial^2 u}{\partial t^2} + \frac{a_1 + a_{-1}}{2} h^2 \frac{\partial^2 u}{\partial x^2} + \mathscr{O}(a^0 r^3 h^3, a_1 h^3, a_{-1} h^3) \tag{10.52}$$

that has order $\mathscr{O}(h)$. Indeed, as the expressions for the coefficients a^0, a_{-1}, a_0, and a_1 all contain the grid size h in the denominator, we conclude that the first two terms in the previous sum have order $\mathscr{O}(h)$, while the quantity $\mathscr{O}(a^0 r^3 h^3, a_1 h^3, a_{-1} h^3)$ is, in fact, of order $\mathscr{O}(h^2)$.

In general, among the smooth functions $u = u(x,t)$ there are obviously polynomials of the second degree, for which the derivatives $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^2 u}{\partial x^2}$ can assume arbitrary independent values at any fixed point (x,t). Moreover, the term $\mathscr{O}(a^0 r^3 h^3, a_1 h^3, a_{-1} h^3)$ that contains the third derivatives of the polynomials vanishes. Therefore, in order to guarantee that the order of the residual (10.52) [remainder of the approximation (10.49)] be at least $\mathscr{O}(h)$, we must require that the coefficients in front of $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^2 u}{\partial x^2}$ both be of order h, independently of one another. On the other hand, from the first equation of (10.50) we always have $a^0 = 1/rh$, and consequently, the coefficient in front of $\frac{\partial^2 u}{\partial t^2}$ in the sum (10.52) is equal to $rh/2$. As such, the order of the residual with respect to the grid size may never exceed $\mathscr{O}(h)$.

Thus, we have established that one cannot construct a consistent scheme $\boldsymbol{L}_h u^{(h)} = f^{(h)}$ of type (10.47) that would approximate the Cauchy problem (10.8) with the accuracy better than $\mathscr{O}(h)$. To achieve, for example, the accuracy of $\mathscr{O}(h^2)$, one would have to use a larger stencil, i.e., employ more grid nodes for building the difference operator $\boldsymbol{L}_h$.

However, under the additional assumption of $\varphi(x,t) \equiv 0$, there exists one and only one scheme $\boldsymbol{L}_h u^{(h)} = f^{(h)}$ of type (10.47) that approximates problem (10.8) with second order of accuracy with respect to h. Let us actually construct this scheme and in doing so also make sure that it is unique.[5] For that purpose, we first notice that the

[5] Uniqueness is to be understood in the same sense as on page 336.