Lagrange form of the interpolating polynomial (2.1) and definition (2.17) it is clear that:

$$L_n = \mathcal{O}\left(\max_{a\le x\le b}\sum_{k=0}^{n}|l_k(x)|\right) \tag{2.19}$$

(later, see Section 3.2.7 of Chapter 3, we will prove an even more precise statement). Take $k \approx n/2$ and x very close to one of the edges a or b, say, $x-a=\eta \ll h$. Then,

$$\begin{aligned}
|l_k(x)| &= \left|\frac{(x-x_0)(x-x_1)\dots(x-x_{k-1})(x-x_{k+1})\dots(x-x_n)}{(x_k-x_0)(x_k-x_1)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_n)}\right| \\
&\approx \frac{\eta\cdot h^{2k-1}\cdot(2k)!/k}{(h^k k!)^2} = \frac{\eta}{h}\cdot\frac{(2k)!}{k(k!)^2} \\
&= \frac{\eta}{h}\cdot\frac{(2\cdot 4\cdot 6\cdot\ldots\cdot 2k)(1\cdot 3\cdot 5\cdot\ldots\cdot(2k-1))}{k(k!)^2} \\
&\approx \frac{\eta}{h}\cdot\frac{(2\cdot 4\cdot 6\cdot\ldots\cdot 2k)^2}{(k!)^2} = \frac{\eta}{h}\cdot\frac{2^{2k}(k!)^2}{(k!)^2} \approx \frac{\eta}{h}\cdot 2^n.
\end{aligned}$$

The foregoing estimate for $|l_k(x)|$, along with the previous formula (2.19), do imply the exponential growth of the Lebesgue constants on uniform (equally spaced) interpolation grids. Let now $a=-1$, $b=1$, and let the interpolation nodes on $[a,b]$ be rather given by the formula:

$$x_j = -\cos\frac{(2j+1)\pi}{2(n+1)}, \quad j=0,1,\dots,n. \tag{2.20}$$

It is possible to show that placing the nodes according to (2.20) guarantees a much better estimate for the Lebesgue constants (again, see Section 3.2.7):

$$L_n \le \frac{2}{\pi}\ln(n+1)+1. \tag{2.21}$$

We therefore conclude that in contradistinction to the previous case (2.18), the Lebesgue constants may, in fact, grow slowly rather than rapidly, as they do on the non-equally spaced nodes (2.20). As such, even the high-degree interpolating polynomials in this case will not be overly sensitive to perturbations of the input data. Interpolation nodes (2.20) are known as the Chebyshev nodes. They will be discussed in detail in Chapter 3.

2.1.5 On Poor Convergence of Interpolation with Equidistant Nodes

One should not think that for any continuous function $f(x)$, $x\in[a,b]$, the algebraic interpolating polynomials $P_n(x,f)$ built on the equidistant nodes $x_j = a + j\cdot h$, $x_0 = a$, $x_n = b$, will converge to $f(x)$ as n increases, i.e., that the deviation of $P_n(x,f)$ from $f(x)$ will decrease. For example, as has been shown by Bernstein, the sequence