Let U and F be two Banach spaces, and let $\boldsymbol{L}$ be a linear operator: $\boldsymbol{L} : U \longmapsto F$ that has a bounded inverse, $\boldsymbol{L}^{-1} : F \longmapsto U$, $\|\boldsymbol{L}^{-1}\| < \infty$. In other words, we assume that the problem

$$\boldsymbol{L}u = f \tag{10.26}$$

is uniquely solvable for every $f \in F$ and well-posed.

Let $\boldsymbol{L}_h : U \longmapsto F$ be a family of operators parameterized by some h (for example, we may have $h = 1/n$, $n = 1, 2, 3, \ldots$). Along with the original problem (10.26), we introduce a series of its "discrete" counterparts:

$$\boldsymbol{L}_h u^{(h)} = f, \tag{10.27}$$

where $u^{(h)} \in U$ and each $\boldsymbol{L}_h$ is also assumed to have a bounded inverse, $\boldsymbol{L}_h^{-1} : F \longmapsto U$, $\|\boldsymbol{L}_h^{-1}\| < \infty$. The operators $\boldsymbol{L}_h$ are referred to as approximating operators.

We say that problem (10.27) is consistent, or in other words, that the operators $\boldsymbol{L}_h$ of (10.27) approximate the operator $\boldsymbol{L}$ of (10.26), if for any $u \in U$ we have

$$\|\boldsymbol{L}_h u - \boldsymbol{L}u\|_F \longrightarrow 0, \quad \text{as} \quad h \longrightarrow 0. \tag{10.28}$$

Note that any given $u \in U$ can be interpreted as solution to problem (10.26) with the right-hand side defined as $F \ni f \stackrel{\text{def}}{=} \boldsymbol{L}u$. Then, the general notion of consistency (10.28) becomes similar to the concept of approximation on a solution introduced in Section 10.1.2, see formula (10.6).

Problem (10.27) is said to be stable if all the inverse operators are bounded uniformly:

$$\|\boldsymbol{L}_h^{-1}\| \leq C = \text{const}, \tag{10.29}$$

which means that C does not depend on h. This is obviously a stricter condition than simply having each $\boldsymbol{L}_h^{-1}$ bounded; it is, again, similar to Definition 10.2 of Section 10.1.3.

THEOREM 10.2 (Kantorovich)

Provided that properties (10.28) and (10.29) hold, the solution $u^{(h)}$ of the approximating problem (10.27) converges to the solution u of the original problem (10.26):

$$\|u - u^{(h)}\|_U \longrightarrow 0, \quad \text{as} \quad h \longrightarrow 0. \tag{10.30}$$

PROOF Given (10.28) and (10.29), we have

$$\begin{aligned}
\|u - u^{(h)}\|_U &= \|\boldsymbol{L}_h^{-1}\boldsymbol{L}_h u - \boldsymbol{L}_h^{-1} f\|_U \leq \|\boldsymbol{L}_h^{-1}\|\|\boldsymbol{L}_h u - f\|_F \\
&\leq C\|\boldsymbol{L}_h u - f\|_F = \leq C\|\boldsymbol{L}_h u - \boldsymbol{L}u + \boldsymbol{L}u - f\|_F \\
&= C\|\boldsymbol{L}_h u - \boldsymbol{L}u\|_F \longrightarrow 0, \quad \text{as} \quad h \longrightarrow 0,
\end{aligned}$$

because $\boldsymbol{L}u = f$ and $\boldsymbol{L}_h u^{(h)} = f$. ∎