By continuing this line of argument, we conclude that the n-th derivative $\varphi^{(n)}(x)$ will have at least one zero on the interval $[x_0,x_n]$. Let us denote this zero by ξ, so that $\varphi^{(n)}(\xi)=0$. Next, we differentiate identity (2.14) exactly n times and subsequently substitute $x=\xi$, which yields:

$$0=\varphi^{(n)}(\xi)=f^{(n)}(\xi)-\frac{d^n}{dx^n}P_n(x,f,x_0,x_1,\dots,x_n)\Big|_{x=\xi}. \tag{2.15}$$

On the other hand, according to Corollary 2.2, the divided difference $f(x_0,x_1,\dots,x_n)$ is equal to the leading coefficient of the interpolating polynomial P_n, i.e., $P_n(x,f,x_0,x_1,\dots,x_n)=f(x_0,x_1,\dots,x_n)x^n+c_{n-1}x^{n-1}+\dots+c_0$. Consequently, $\frac{d^n}{dx^n}P_n(x,f,x_0,x_1,\dots,x_n)=n!f(x_0,x_1,\dots,x_n)$, and therefore, equality (2.15) implies (2.13). $\square$

THEOREM 2.4
The values $f(x_0), f(x_1), \dots, f(x_n)$ of the function $f(x)$ are expressed through the divided differences $f(x_0)$, $f(x_0,x_1),\dots,$ $f(x_0,x_1,\dots,x_n)$ by the formulae:

$$\begin{aligned}f(x_j)=&f(x_0)+(x_j-x_0)f(x_0,x_1)+(x_j-x_0)(x_j-x_1)f(x_0,x_1,x_2)+\dots\\&+(x_j-x_0)(x_j-x_1)\dots(x_j-x_{n-1})f(x_0,x_1,\dots,x_n),\qquad j=0,1,\dots,n,\end{aligned}$$

i.e., by linear combinations of the type:

$$f(x_j)=a_{j0}f(x_0)+a_{j1}f(x_0,x_1)+\dots+a_{jn}f(x_0,x_1,\dots,x_n),\qquad j=0,1,\dots,n. \tag{2.16}$$

PROOF The result follows immediately from formula (2.3) and equalities $f(x_j)=P(x,f,x_0,x_1,\dots,x_n)\big|_{x=x_j}$ for $j=0,1,\dots,n$. $\square$

2.1.3 Comparison of the Lagrange and Newton Forms

To evaluate the function $f(x)$ at a point x that is not one of the interpolation nodes, one can approximately set: $f(x)\approx P_n(x,f,x_0,x_1,\dots,x_n)$.

Assume that the polynomial $P_n(x,f,x_0,x_1,\dots,x_n)$ has already been built, but in order to try and improve the accuracy we incorporate an additional interpolation node x_{n+1} and the corresponding function value $f(x_{n+1})$. Then, to construct the interpolating polynomial $P_{n+1}(x,f,x_0,x_1,\dots,x_{n+1})$ using the Lagrange formula (2.1) one basically needs to start from the scratch. At the same time, to use the Newton formula (2.3), see also Corollary 2.1:

$$\begin{aligned}P_{n+1}(x,f,x_0,x_1,\dots,x_{n+1})=&P_n(x,f,x_0,x_1,\dots,x_n)\\&+(x-x_0)(x-x_1)\dots(x-x_n)f(x_0,x_1,\dots,x_{n+1})\end{aligned}$$

one only needs to obtain the correction

$$(x-x_0)(x-x_1)\dots(x-x_n)f(x_0,x_1,\dots,x_{n+1}).$$

Moreover, one will immediately be able to see how large this correction is.