PROOF As always, we denote the error of the iterate $x^{(p)}$ by $\varepsilon^{(p)} = \hat{x} - x^{(p)}$. Then, we can write:

$$0 = F(\hat{x}) = F(x^{(p)} + \varepsilon^{(p)}) = F(x^{(p)}) + F'(x^{(p)})\varepsilon^{(p)} + \frac{1}{2}F''(\xi)(\varepsilon^{(p)})^2,$$

where ξ is some intermediate point between $\hat{x}$ and $x^{(p)}$. Consequently,

$$\varepsilon^{(p)} = \hat{x} - x^{(p)} = -\frac{1}{F'(x^{(p)})}\left[F(x^{(p)}) + \frac{1}{2}F''(\xi)(\varepsilon^{(p)})^2\right].$$

On the other hand, according to the definition of Newton's method, see formula (8.11), we have:

$$x^{(p+1)} - x^{(p)} = -\frac{F(x^{(p)})}{F'(x^{(p)})},$$

which, after the substitution into the previous formula, yields:

$$\varepsilon^{(p+1)} = \hat{x} - x^{(p+1)} = -\frac{1}{2}\frac{F''(\xi)}{F'(x^{(p)})}(\varepsilon^{(p)})^2.$$

By the hypothesis of the theorem, on some neighborhood of the root $\hat{x}$ we have:

$$\frac{1}{|F'(x)|} \leq C_1 \quad \text{and} \quad |F''(x)| \leq C_2,$$

where C_1 and C_2 are two constants. Therefore,

$$|\varepsilon^{(p+1)}| = |\hat{x} - x^{(p+1)}| \leq \frac{1}{2}C_1C_2|\varepsilon^{(p)}|^2 = \frac{1}{2}C_1C_2|\hat{x} - x^{(p)}|^2,$$

which implies quadratic convergence. $\square$

8.3.2 Newton's Linearization for Systems

Similarly to the scalar case, Newton's method can also be applied to solving systems of nonlinear equations $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{0}$, where $\boldsymbol{F}$ is a mapping, $\boldsymbol{F} : \mathbb{R}^n \longmapsto \mathbb{R}^n$. Hereafter, we will assume that $\boldsymbol{F}(\boldsymbol{x})$ is continuously differentiable on some domain $\Omega \subseteq \mathbb{R}^n$ that contains the desired solution $\hat{\boldsymbol{x}}$: $\boldsymbol{F}(\hat{\boldsymbol{x}}) = \boldsymbol{0}$. Then, for any $\boldsymbol{x}^{(p)}$, $p = 0, 1, 2, \ldots$, the Taylor-based linearization yields:

$$\boldsymbol{F}(\boldsymbol{x}) \approx \boldsymbol{F}(\boldsymbol{x}^{(p)}) + \frac{\partial \boldsymbol{F}(\boldsymbol{x}^{(p)})}{\partial \boldsymbol{x}}(\boldsymbol{x} - \boldsymbol{x}^{(p)}),$$

where $\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}$ is the Jacobi matrix, or Jacobian, of the mapping $\boldsymbol{F}$:

$$\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}} \stackrel{\text{def}}{=} \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix} \equiv \boldsymbol{J_F}.$$