If the norm of this matrix is bounded by some number q, $0 \le q < 1$, for all $\boldsymbol{x} \in \Omega$, then the mapping $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x})$ is a contraction on Ω, i.e., the following inequality holds:

$$\|\boldsymbol{f}(\boldsymbol{x}') - \boldsymbol{f}(\boldsymbol{x}'')\| \le q\|\boldsymbol{x}' - \boldsymbol{x}''\|,$$

where $\boldsymbol{x}'$ and $\boldsymbol{x}''$ are two arbitrary points from Ω.

The proof of this theorem is the subject of Exercise 1.

To actually use the fixed point iterations (8.10) for computing an approximation to the solution $\hat{\boldsymbol{x}}$ of the equation $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{0}$, one needs to exploit the flexibility that exists in choosing the auxiliary mapping $\boldsymbol{f}(\boldsymbol{x})$ so as to make it a contraction with the smallest possible coefficient q. This will guarantee the fastest convergence.

REMARK 8.2 In Section 6.1, we solved the linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ by first reducing it to an equivalent form $\boldsymbol{x} = \boldsymbol{B}\boldsymbol{x} + \boldsymbol{\varphi}$ and then employing the iteration $\boldsymbol{x}^{(p+1)} = \boldsymbol{B}\boldsymbol{x}^{(p)} + \boldsymbol{\varphi}$, $p = 0, 1, 2, \ldots$. In the context of this section, we can define $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{B}\boldsymbol{x} + \boldsymbol{\varphi}$, in which case the Jacobi matrix $\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = \boldsymbol{B}$. Then, according to Theorem 8.4, if $\|\boldsymbol{B}\| = q < 1$ then the mapping $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x})$ is a contraction. This, in turn, guarantees convergence of the fixed point iteration (8.10). As such, we see that Theorem 6.1 (see page 175) that guarantees convergence of the linear iteration provided that $\|\boldsymbol{B}\| = q < 1$, can be considered a direct implication of the results of this section in the case of linear systems. $\square$

Exercises

1.* Prove Theorem 8.4.

Hint. Represent the increment of the function $\boldsymbol{f}(\boldsymbol{x})$ as the integral of the derivative in the direction $\boldsymbol{x}' - \boldsymbol{x}''$.

8.3 Newton's Method

8.3.1 Newton's Linearization for One Scalar Equation

As shown in Section 8.1.4, Newton's method for finding the solution of the nonlinear equation $F(x) = 0$ is based on linearization of the function $F(x)$. Let $x^{(0)}$ be the initial guess, and let $x^{(p)}$ be the current iterate. For any given $p = 0, 1, 2, \ldots$ we can write the following approximate linear formula:

$$F(x) \approx F(x^{(p)}) + F'(x^{(p)})(x - x^{(p)}),$$