A minimum norm weak (generalized) solution of the overdetermined system (7.27) is the vector $\hat{\boldsymbol{x}} \in \mathbb{R}^n$ *that minimizes* $\Phi(\boldsymbol{x})$*, i.e.,* $\forall \boldsymbol{x} \in \mathbb{R}^n$: $\Phi(\boldsymbol{x}) \geq \Phi(\hat{\boldsymbol{x}})$*, and also such that if there is another* $\boldsymbol{x} \in \mathbb{R}^n$, $\Phi(\boldsymbol{x}) = \Phi(\hat{\boldsymbol{x}})$*, then* $\|\boldsymbol{x}\|_2 \geq \|\hat{\boldsymbol{x}}\|_2$.

Note that the minimum norm weak solution introduced according to Definition 7.3 may exhibit strong sensitivity to the perturbations of the matrix $\boldsymbol{A}$ in the case when these perturbations change the rank of the matrix, see the example given in Exercise 2 after the section.

REMARK 7.6 Definition 7.3 can also be applied to the case of a full rank matrix $\boldsymbol{A}$, $\operatorname{rank}\boldsymbol{A} = n$. Then it reduces to Definition 7.1 (for $\boldsymbol{B} = \boldsymbol{I}$), because according to Theorem 7.2 a unique least squares weak solution exists for a full rank overdetermined system, and consequently, the Euclidean norm of this solution is minimum. □

THEOREM 7.3

Let $\boldsymbol{A}$ *be an* $m \times n$ *matrix with real entries,* $m \geq n$*, and let* $\operatorname{rank}\boldsymbol{A} = r < n$*. There is a unique weak solution of system (7.27) in the sense of Definition 7.3. This solution is given by the formula:*

$$\hat{\boldsymbol{x}} = \boldsymbol{A}^{+}\boldsymbol{f}, \tag{7.33}$$

where $\boldsymbol{A}^{+}$ *is the Moore-Penrose pseudoinverse of* $\boldsymbol{A}$ *introduced in Definition 7.2.*

PROOF Using singular value decomposition, represent the system matrix of (7.27) in the form: $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{W}^*$. Also define $\boldsymbol{y} = \boldsymbol{W}^*\boldsymbol{x}$. Then, according to formula (7.32), we can write:

$$\begin{aligned}\Phi(\boldsymbol{x}) &= (\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{W}^*\boldsymbol{x} - \boldsymbol{f}, \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{W}^*\boldsymbol{x} - \boldsymbol{f})^{(m)} = (\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{y} - \boldsymbol{f}, \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{y} - \boldsymbol{f})^{(m)} \\ &= (\boldsymbol{\Sigma}\boldsymbol{y} - \boldsymbol{U}^*\boldsymbol{f}, \boldsymbol{\Sigma}\boldsymbol{y} - \boldsymbol{U}^*\boldsymbol{f})^{(m)} = \|\boldsymbol{\Sigma}\boldsymbol{y} - \boldsymbol{U}^*\boldsymbol{f}\|_2^2,\end{aligned}$$

and we need to find the vector $\hat{\boldsymbol{y}} \in \mathbb{R}^n$ such that $\forall \boldsymbol{y} \in \mathbb{R}^n$: $\|\boldsymbol{\Sigma}\hat{\boldsymbol{y}} - \boldsymbol{U}^*\boldsymbol{f}\|_2^2 \leq \|\boldsymbol{\Sigma}\boldsymbol{y} - \boldsymbol{U}^*\boldsymbol{f}\|_2^2$. This vector $\hat{\boldsymbol{y}}$ must also have a minimum Euclidean norm, because the matrix $\boldsymbol{W}$ is orthogonal and since $\boldsymbol{y} = \boldsymbol{W}^*\boldsymbol{x}$ we have $\|\boldsymbol{y}\|_2 = \|\boldsymbol{x}\|_2$.

Next, recall that as $\operatorname{rank}\boldsymbol{A} = r$, the matrix $\boldsymbol{A}$ has precisely r non-zero singular values σ_i. Then we have:

$$\|\boldsymbol{\Sigma}\boldsymbol{y} - \boldsymbol{U}^*\boldsymbol{f}\|_2^2 = \sum_{i=1}^{r} [\sigma_i y_i - (\boldsymbol{U}^*\boldsymbol{f})_i]^2 + \sum_{i=r+1}^{m} [(\boldsymbol{U}^*\boldsymbol{f})_i]^2, \tag{7.34}$$

where $(\boldsymbol{U}^*\boldsymbol{f})_i$ denotes component number i of the m-dimensional vector $\boldsymbol{U}^*\boldsymbol{f}$. Expression (7.34) attains its minimum value when the first sum on the right-hand side is equal to zero, because the second sum simply does not depend