

the method guarantees that the number of iterations required to reduce the initial error by a prescribed factor will only be proportional to the square root of the condition number $\sqrt{\mu(\mathbf{A})}$, see formula (6.65).

The key shortcoming of the method of conjugate gradients is its computational instability that manifests itself stronger for larger condition numbers $\mu(\mathbf{A})$. The most favorable situation for applying the method of conjugate gradients is when it is known that the condition number $\mu(\mathbf{A})$ is not too large, while the boundaries of the spectrum are unknown, and the dimension n of the system is much higher than the number of iterations p needed for achieving a given accuracy. In practice, one can also use special stabilization procedures for the method of conjugate gradients, as well as implement restarts after every so many iterations. The latter, however, slow down the convergence.

Note also that both the Chebyshev iterative method and the method of conjugate gradients can be preconditioned, similarly to how the Richardson method was preconditioned in Section 6.1.4. Preconditioning helps reduce the condition number and as such, speeds up the convergence.

Exercises

1. Consider the same second order central difference scheme as the one built in Exercise 4 after Section 5.4 (page 156):

$$\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} - u_j = f_j, \quad j = 1, 2, \dots, N-1,$$

$$h = \frac{1}{N}, \quad u_0 = 0, \quad u_N = 0.$$

- a) Write down this scheme as a system of linear algebraic equations with an $(N-1) \times (N-1)$ matrix $\mathbf{A}$. Use the apparatus of finite Fourier series (Section 5.7) to prove that the matrix $-\mathbf{A}$ is symmetric positive definite: $-\mathbf{A} = -\mathbf{A}^* > 0$, find sharp boundaries of its spectrum $\lambda_{\min}$ and $\lambda_{\max}$, and estimate its Euclidean condition number $\mu(-\mathbf{A})$ as it depends on $h = 1/N$ for large N .

Hint. The eigenvectors and eigenvalues of $-\mathbf{A}$ are given by $\psi_j^{(k)} = \sqrt{2} \sin \frac{k\pi j}{N}$, $j, k = 1, 2, \dots, N-1$, and $\lambda_k = \left(\frac{4}{h^2} \sin^2 \frac{k\pi}{2N} + 1 \right)$, $k = 1, 2, \dots, N-1$, respectively.

- b) Solve the system

$$-\mathbf{A}\mathbf{u} = -\mathbf{f}$$

on the computer by iterations, with the initial guess $\mathbf{u}^{(0)} = \mathbf{0}$ and the same right-hand side $\mathbf{f}$ as in Exercise 4 after Section 5.4: $f_j = f(x_j)$, $j = 1, \dots, N-1$, where $f(x) = (-\pi^2 \sin(\pi x) + 2\pi \cos(\pi x))e^x$ and $x_j = j \cdot h$; the corresponding values of the exact solution are $u(x_j)$, $j = 1, \dots, N-1$, where $u(x) = \sin(\pi x)e^x$.

Implement three different iteration schemes: The non-preconditioned stationary Richardson method of Section 6.1.3, the Chebyshev method of Section 6.2.1, and the conjugate gradients method of Section 6.2.2. For each scheme, use four different grids: $N = 32, 64, 128$, and 256 . Evaluate the error $\mathbf{u}_{\text{exact}} - \mathbf{u}^{(p)}$ in the Euclidean norm based on the inner product (5.98): $(\mathbf{v}, \mathbf{w}) = h \sum_{j=1}^{N-1} v_j w_j$ (recall,