To find the initial guess that would turn (6.26) into an equality, we first introduce our standard notation $\boldsymbol{\varepsilon}^{(p)} = \boldsymbol{x} - \boldsymbol{x}^{(p)}$ for the error of the iterate $\boldsymbol{x}^{(p)}$, and subtract equation (6.25) from $\boldsymbol{x} = \boldsymbol{B}\boldsymbol{x} + \boldsymbol{\varphi}$, which yields: $\boldsymbol{\varepsilon}^{(p+1)} = \boldsymbol{B}\boldsymbol{\varepsilon}^{(p)}$, $p = 0, 1, 2, \ldots$. Next, suppose that $|\nu_k| = \max_j |\nu_j| = \rho$ and take $\boldsymbol{\varepsilon}^{(0)} = \boldsymbol{x} - \boldsymbol{x}^{(0)} = \boldsymbol{e}_k$, where $\boldsymbol{e}_k$ is the eigenvector of $\boldsymbol{B}$ that corresponds to the eigenvalue with maximum magnitude. Then we obtain: $\|\boldsymbol{\varepsilon}^{(p)}\| = |\nu_k|^p \|\boldsymbol{\varepsilon}^{(0)}\| = \rho^p \|\boldsymbol{\varepsilon}^{(0)}\|$.

To prove the second conclusion of the lemma, we take the particular eigenvalue ν_k that delivers the maximum: $|\nu_k| = \max_j |\nu_j| = \rho \geq 1$, and again select $\boldsymbol{\varepsilon}^{(0)} = \boldsymbol{x} - \boldsymbol{x}^{(0)} = \boldsymbol{e}_k$, where $\boldsymbol{e}_k$ is the corresponding eigenvector. In this case the error obviously does not vanish as $p \longrightarrow \infty$, because:

$$\boldsymbol{\varepsilon}^{(p)} = \boldsymbol{B}\boldsymbol{\varepsilon}^{(p-1)} = \ldots = \boldsymbol{B}^p \boldsymbol{\varepsilon}^{(0)} = \nu_k^p \boldsymbol{e}_k,$$

and consequently, $\|\boldsymbol{\varepsilon}^{(p)}\| = \rho^p \|\boldsymbol{e}_k\|$, where ρ^p will either stay bounded but will not vanish, or will increase when $p \longrightarrow \infty$. $\square$

Lemma 6.1 analyzes a special case $\boldsymbol{B} = \boldsymbol{B}^*$ and provides a simple illustration for the general conclusion of Theorem 6.2 that for the convergence of a first order linear stationary iteration it is necessary and sufficient that the spectral radius of the iteration matrix be strictly less than one. With the help of this lemma, we will now analyze convergence of the stationary Richardson iteration (6.5) for the case $\boldsymbol{A} = \boldsymbol{A}^* > 0$.

THEOREM 6.3

Consider a system of linear algebraic equations:

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{f}, \quad \boldsymbol{A} = \boldsymbol{A}^* > 0, \tag{6.27}$$

where $\boldsymbol{x}, \boldsymbol{f} \in \mathbb{L}$, *and* $\mathbb{L}$ *is an* n*-dimensional Euclidean space (e.g.,* $\mathbb{L} = \mathbb{R}^n$*). Let* $\lambda_{\min}$ *and* $\lambda_{\max}$ *be the smallest and the largest eigenvalues of the operator* $\boldsymbol{A}$*, respectively. Specify some* $\tau \neq 0$ *and recast system (6.27) in an equivalent form:*

$$\boldsymbol{x} = (\boldsymbol{I} - \tau \boldsymbol{A})\boldsymbol{x} + \tau \boldsymbol{f}. \tag{6.28}$$

Given an arbitrary initial guess $\boldsymbol{x}^{(0)} \in \mathbb{L}$*, consider the sequence of Richardson iterations:*

$$\boldsymbol{x}^{(p+1)} = (\boldsymbol{I} - \tau \boldsymbol{A})\boldsymbol{x}^{(p)} + \tau \boldsymbol{f}, \quad p = 0, 1, 2, \ldots. \tag{6.29}$$

1. *If the parameter* τ *satisfies the inequalities:*

$$0 < \tau < \frac{2}{\lambda_{\max}}, \tag{6.30}$$

then the sequence $\boldsymbol{x}^{(p)}$ *of (6.29) converges to the solution* $\boldsymbol{x}$ *of system (6.27). Moreover, the norm of the error* $\|\boldsymbol{x} - \boldsymbol{x}^{(p)}\|$ *is guaranteed to decrease when* p *increases with the rate given by the following estimate:*

$$\|\boldsymbol{x} - \boldsymbol{x}^{(p)}\| \leq \rho^p \|\boldsymbol{x} - \boldsymbol{x}^{(0)}\|, \quad p = 0, 1, 2, \ldots. \tag{6.31}$$