on a real machine with a fixed number of significant digits, along with the effect of inaccuracies in the input data, round-off errors are introduced at every arithmetic operation. For linear systems, the impact of these round-off errors depends on the number of significant digits, on the condition number of the matrix, as well as on the particular algorithm chosen for computing the solution. In work [GAKK93], a family of algorithms has been proposed that directly take into account the effect of rounding on a given computer. These algorithms either produce the result with a guaranteed accuracy, or otherwise determine in the course of computations that the system is conditioned so poorly that no accuracy can be guaranteed when computing its solution on the machine with a given number of significant digits.

Exercises

1. Write explicitly the matrices $\tilde{\boldsymbol{T}}_k^{-1}$ and $\tilde{\boldsymbol{L}}$ that correspond to the $\boldsymbol{LU}$ decomposition of $\boldsymbol{A}$ with $\tilde{\boldsymbol{T}}_k$ and $\tilde{\boldsymbol{U}}$ defined by formulae (5.76) and (5.77), respectively.

2. Compute the solution of the 2×2 system:

$$10^{-3}x+y=5, \quad x-y=6,$$

using standard Gaussian elimination and Gaussian elimination with pivoting. Conduct all computations with two significant digits (decimal). Compare and explain the results.

3.* Show that when performing Gaussian elimination with partial pivoting, the $\boldsymbol{LU}$ factorization is obtained in the form:

$$\boldsymbol{PA}=\boldsymbol{LU},$$

where $\boldsymbol{P}$ is the composition (i.e., product) of all permutation matrices used at every stage of the algorithm.

4. Consider a boundary value problem for the second order ordinary differential equation:

$$\frac{d^2u}{dx^2}-u=f(x), \quad x\in[0,1],$$
$$u(0)=0, \;\; u(1)=0,$$

where $u=u(x)$ is the unknown function and $f=f(x)$ is a given right-hand side. To solve this problem numerically, we first partition the interval $0\le x\le 1$ into N equal subintervals and thus build a uniform grid of $N+1$ nodes: $x_j=j\cdot h$, $h=1/N$, $j=0,1,2,\dots,N$. Then, instead of looking for the continuous function $u=u(x)$ we will be looking for its approximate table of values $\{u_0,u_1,u_2,\dots,u_N\}$ at the grid nodes $x_0,x_1,x_2,\dots,x_N$, respectively.

At every interior node x_j, $j=1,2,\dots,N-1$, we approximately replace the second derivative by the difference quotient (for more detail, see Section 9.2):

$$\left.\frac{d^2u}{dx^2}\right|_{x=x_j}\approx\frac{u_{j+1}-2u_j+u_{j-1}}{h^2},$$

and arrive at the following finite-difference counterpart of the original problem (a central-difference scheme):

$$\frac{u_{j+1}-2u_j+u_{j-1}}{h^2}-u_j=f_j, \quad j=1,2,\dots,N-1,$$
$$u_0=0, \;\; u_N=0,$$