5.4.3 Cyclic Tri-Diagonal Elimination

In many applications, one needs to solve a system which is "almost" tri-diagonal, but is not quite equivalent to system (5.59):

$$
\begin{array}{l}
b_1x_1 + c_1x_2 \qquad\qquad\qquad\qquad + a_1x_n = f_1, \\
a_2x_1 + b_2x_2 + c_2x_3 \qquad\qquad\qquad = f_2, \\
\qquad a_3x_2 + b_3x_3 + c_3x_4 \qquad\qquad = f_3, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\qquad\qquad a_{n-1}x_{n-2} + b_{n-1}x_{n-1} + c_{n-1}x_n = f_{n-1}, \\
c_nx_1 \qquad\qquad\qquad + a_nx_{n-1} + b_nx_n = f_n.
\end{array}
\tag{5.66}
$$

In Section 2.3.2, we have analyzed one particular example of this type that arises when constructing the nonlocal Schoenberg splines; see the matrix $\boldsymbol{A}$ given by formula (2.66) on page 52. Other typical examples include the so-called central difference schemes (see Section 9.2.1) for the solution of second order ordinary differential equations with periodic boundary conditions, as well as many schemes built for solving partial differential equations in the cylindrical or spherical coordinates. Periodicity of the boundary conditions gave rise to the name *cyclic* attached to the version of the tri-diagonal elimination that we are about to describe.

The coefficients a_1 and c_n in the first and last equations of system (5.66), respectively, are, generally speaking, non-zero. Their presence does not allow one to apply the tri-diagonal elimination algorithm of Section 5.4.2 to system (5.66) directly. Let us therefore consider two auxiliary linear systems of dimension $(n-1)\times(n-1)$:

$$
\begin{array}{l}
b_2u_2 + c_2u_3 \qquad\qquad\qquad = f_2, \\
a_3u_2 + b_3u_3 + c_3u_4 \qquad\quad = f_3, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\qquad a_{n-1}u_{n-2} + b_{n-1}u_{n-1} + c_{n-1}u_n = f_{n-1}, \\
\qquad\qquad\qquad a_nu_{n-1} + b_nu_n = f_n.
\end{array}
\tag{5.67}
$$

and

$$
\begin{array}{l}
b_2v_2 + c_2v_3 \qquad\qquad\qquad = -a_2, \\
a_3v_2 + b_3v_3 + c_3v_4 \qquad\quad = 0, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\qquad a_{n-1}v_{n-2} + b_{n-1}v_{n-1} + c_{n-1}v_n = 0, \\
\qquad\qquad\qquad a_nv_{n-1} + b_nv_n = -c_n.
\end{array}
\tag{5.68}
$$

Having obtained the solutions $\{u_2,u_3,\dots,u_n\}$ and $\{v_2,v_3,\dots,v_n\}$ to systems (5.67) and (5.68), respectively, we can represent the solution $\{x_1,x_2,\dots,x_n\}$ to system (5.66) in the form:

$$x_i = u_i + x_1v_i, \quad i = 1,2,\dots,n, \tag{5.69}$$

where for convenience we additionally define $u_1 = 0$ and $v_1 = 1$. Indeed, multiplying each equation of system (5.68) by x_1, adding with the corresponding equation of system (5.67), and using representation (5.69), we immediately see that the equations number 2 through n of system (5.66) are satisfied. It only remains to satisfy equation number 1 of system (5.66). To do so, we use formula (5.69) for $i = 2$ and $i = n$, and