Generaly speaking, the linear space $\mathbb{L}$ is complex and the scalar product $(\boldsymbol{x},\boldsymbol{y})$ is a complex number unless $\boldsymbol{x}=\boldsymbol{y}$, in which case it is real (axiom 4). If, however, the space $\mathbb{L}$ is real, then axioms 2, 3, and 4 remain unchanged, whereas axiom 1 becomes $(\boldsymbol{x},\boldsymbol{y})=(\boldsymbol{y},\boldsymbol{x})$, i.e., the scalar product commutative. Using axioms 1 – 4, one can also prove the Cauchy-Schwartz inequality that holds for any $\boldsymbol{x}$, $\boldsymbol{y}\in\mathbb{L}$:

$$|(\boldsymbol{x},\boldsymbol{y})|^2 \le (\boldsymbol{x},\boldsymbol{x})(\boldsymbol{y},\boldsymbol{y}).$$

The simplest example of a scalar product on the space $\mathbb{R}^n$ is given by

$$(\boldsymbol{x},\boldsymbol{y}) = x_1y_1 + x_2y_2 + \ldots + x_ny_n, \tag{5.15}$$

whereas for the complex space $\mathbb{C}^n$ we can choose:

$$(\boldsymbol{x},\boldsymbol{y}) = x_1\bar{y}_1 + x_2\bar{y}_2 + \ldots + x_n\bar{y}_n. \tag{5.16}$$

Recall that a real linear space equipped with an inner product is called *Euclidean,* whereas a complex space with an inner product is called *unitary.*

One can show that the function:

$$\|\boldsymbol{x}\|_2 = (\boldsymbol{x},\boldsymbol{x})^{\frac{1}{2}} \tag{5.17}$$

provides a norm on both $\mathbb{R}^n$ and $\mathbb{C}^n$. In the case of a real space this norm is called *Euclidean*, and in the case of a complex space it is called *Hermitian*. In the literature, the norm defined by formula (5.17) is also referred to as the l_2 norm.

In the previous examples, we have employed a standard vector form $\boldsymbol{x} = [x_1,x_2,\ldots,x_n]^T$ for the elements of the space $\mathbb{L}$ ($\mathbb{L}=\mathbb{R}^n$ or $\mathbb{L}=\mathbb{C}^n$). In much the same way, norms can be introduced on linear spaces without having to enumerate consecutively all the components of every vector. Consider, for example, the space $U^{(h)}=\mathbb{R}^n$, $n=(M-1)^2$, of the grid functions $u^{(h)}=\{u_{m_1,m_2}\}$, $m_1,m_2=1,2,\ldots,M-1$, that we have introduced and exploited in Section 5.1.3 for the construction of a finite-difference scheme for the Poisson equation. The maximum norm and the l_1 norm can be introduced on this space as follows:

$$\|u^{(h)}\|_\infty = \max_{m_1,m_2} |u_{m_1,m_2}|, \tag{5.18}$$

$$\|u^{(h)}\|_1 = \sum_{m_1,m_2=1}^{M-1} |u_{m_1,m_2}|. \tag{5.19}$$

Moreover, a scalar product $(u^{(h)},v^{(h)})$ can be introduced in the real linear space $U^{(h)}$ according to the formula [cf. formula (5.15)]:

$$(u^{(h)},v^{(h)}) = h^2 \sum_{m_1,m_2=1}^{M-1} u_{m_1,m_2}v_{m_1,m_2}. \tag{5.20}$$

Then the corresponding Euclidean norm is defined as follows:

$$\|u^{(h)}\|_2 = (u^{(h)},u^{(h)})^{\frac{1}{2}} = \left[h^2 \sum_{m_1,m_2=1}^{M-1} |u_{m_1,m_2}|^2\right]^{\frac{1}{2}}. \tag{5.21}$$