where $\beta_0 = \beta_n = 1/2$ and $\beta_m = 1$ for $m = 1, 2, \ldots, n-1$.

4.3 Improper Integrals. Combination of Numerical and Analytical Methods

Even for the simplest improper integrals, a direct application of the quadrature formulae may encounter serious difficulties. For example, the trapezoidal rule (4.4):

$$\int_a^b f(x)dx \approx \frac{b-a}{n}\left(\frac{f_0}{2} + f_1 + \ldots + f_{n-1} + \frac{f_n}{2}\right)$$

will fail for the case $a = 0$, $b = 1$, and $f(x) = \cos x/\sqrt{x}$, because $f(x)$ has a singularity at $x = 0$ and consequently, $f_0 = f(0)$ is not defined. Likewise, the Simpson formula (4.13) will fail. For the Gaussian quadrature (4.23), the situation may seem a little better at a first glance, because the Chebyshev-Gauss nodes (4.17) do not include the endpoints of the interval. However, the unboundedness of the function and its derivatives will still prevent one from obtaining any reasonable error estimates. At the same time, the integral itself: $\int_0^1 \frac{\cos x}{\sqrt{x}}dx$, obviously exists, and procedures for its efficient numerical approximation need to be developed.

To address the difficulties that arise when computing the values of improper integrals, it is natural to try and employ a combination of analytical and numerical techniques. The role of the analytical part is to reduce the original problem to a new problem that would only require one to evaluate the integral of a smooth and bounded function. The latter can then be done on a computer with the help of a quadrature formula. In the previous example, we can first use the integration by parts:

$$\int_0^1 \frac{\cos x}{\sqrt{x}}dx = \cos x(2\sqrt{x})\Big|_0^1 + \int_0^1 2\sqrt{x}\sin x dx,$$

and subsequently approximate the integral on the right-hand side to a desired accuracy using any of the quadrature formulae introduced in Section 4.1 or 4.2.

An alternative approach is based on splitting the original integral into two:

$$\int_0^1 \frac{\cos x}{\sqrt{x}}dx = \int_0^c \frac{\cos x}{\sqrt{x}}dx + \int_c^1 \frac{\cos x}{\sqrt{x}}dx, \tag{4.28}$$

where $c > 0$ can be considered arbitrary as of yet. To evaluate the first integral on the right-hand side of equality (4.28), we can use the Taylor expansion:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots.$$