A Generalized Fifth Order WENO Finite Difference Scheme with Z-Type Nonlinear Weights

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2. The WENO-Z Type Nonlinear Weight
3. The Generalized WENO Scheme with Z-Type Nonlinear Weights
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    - Discontinuous Function: Essentially Non-Oscillatory
4. Numerical Examples
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High Order WENO Finite Difference Scheme

Consider the hyperbolic conservation laws

$$\frac{\partial \mathbf{Q}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{Q}) = 0.$$

The semi-discretized of the equation, by method of lines, on a uniformly sized cell, in a conservative manner

$$\frac{d\bar{Q}_i(t)}{dt} = \frac{1}{\Delta x}\left(h_{i+\frac{1}{2}} - h_{i-\frac{1}{2}}\right), \qquad h = h(\bar{Q}_{i-r}, \cdots, \bar{Q}_{i+l}).$$

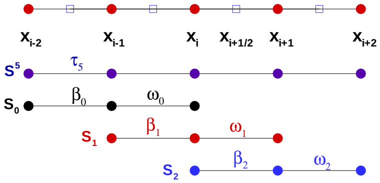
where $h(x)$ is defined implicitly as $f(x) = \frac{1}{\Delta x}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} h(\xi)d\xi$.

Nonlinear spatial adaptive combination of **THREE** Lagrange polynomials $q^k(x)$ of degree 2 in S_k, where $k = 0, 1, 2$ is the shift parameter,

$$\hat{f}(x) = \sum_{k=0}^{2} \omega_k \, q^k(x) \approx h(x) + O(\Delta x^M) \qquad (1)$$

at $x_{i \pm \frac{1}{2}}$, such that, when the solution is

- **SMOOTH**, becomes a $M = 5$ **order central upwinded** scheme.
- **NON-SMOOTH**, becomes a $M = 3$ **order Upwinded** scheme by assigning the nonlinear weight $\omega_k \approx 0$ in S_k containing discontinuity $\Longrightarrow$ **essentially no Gibbs oscillations**.



The nonlinear weights of the classical WENO-JS scheme (Jiang and Shu) are

$$\alpha_k = \frac{d_k}{(\beta_k + \varepsilon)^p}, \qquad \omega_k = \frac{\alpha_k}{\sum_{l=0}^{2} \alpha_l},$$

with two user defined parameters : (1) power parameter $p \geq 1$ and (2) the sensitivity parameter $\varepsilon > 0$ (Usually a fixed small real number).

The **lower order local smoothness indicators**

$$\beta_k = \sum_{l=1}^{2} \Delta x^{2l-1} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(\frac{d^l}{dx^l} q^k(x) \right)^2 dx. \qquad (2)$$

measure the normalized modified Sobolov norm of the second degree polynomials $q^k(x)$ in the substencil S_k at x_i in the cell $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$.

The Improved WENO-Z Scheme

In the $(2r-1)$ order WENO scheme with Z-type weights (WENO-Z), the nonlinear weights are

$$\alpha_k = d_k \left(1 + \left(\frac{\tau_{2r-1}}{\beta_k + \varepsilon}\right)^p\right), \quad \omega_k^Z = \frac{\alpha_k}{\sum_{j=0}^{r-1} \alpha_j}, \quad k = 0, \ldots, r-1.$$

where the global smoothness indicator is

$$\tau_{2r-1} = \left|\sum_{k=0}^{r-1} c_k \beta_k\right|,$$

where c_k are given constants[1]. For example, τ_5 of the fifth order WENO-Z scheme is

$$\tau_5 = |\beta_0 - \beta_2|.$$

Its leading truncation error has been shown to be $O(\Delta x^5)$.

[1] Castro. Costa. and Don. J. Comput. Phys. 230, 1766–1792, 2011

Definition of Critical Points

Definition

If a function $f(x_c) = f'(x_c) = \ldots = f^{(n_{\text{cp}})}(x_c) = 0$ but $f^{(n_{\text{cp}}+1)}(x_c) \neq 0$, the function $f(x)$ is said to have a critical point of order $n_{\text{cp}} \geq 0$ at x_c.

For example, $f(x) = x^3, f'(0) = f''(0) = 0, f'''(0) \neq 0$, then $f(x)$ has $n_{cp} = 2$ at $x = 0$.

$$\alpha_k = d_k \left(1 + \left(\frac{\tau_{2r-1}}{\beta_k + \varepsilon}\right)^p\right).$$

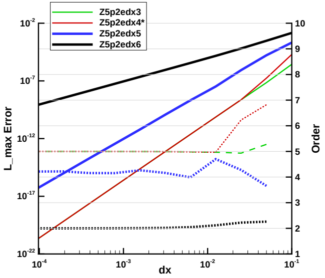
The nonlinear weights α_k have two important free parameters:

- Power p: increases the separation of scales, and controls the amount of numerical dissipation.
- Sensitivity ε: avoids a division by zero in the denominator of α_k.

The issue of critical points

- In general, a very small ε, say $O(10^{-40})$, is highly desirable for capturing shock in an essentially non-oscillatory manner because
  - ε does not over-dominate over the size of the local smoothness indicators β_k as in $(\beta_k + \varepsilon)$.
- However, a very small ε could reduce the formal order of accuracy of WENO schemes of a smooth function in the presence of high order critical points.

$$n_{cp} = 2$$



Optimal Order At Critical Point

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- setting the lower bound on the sensitivity parameters ε by Don et al. (J. Comput. Phys. 250, 2013).

The WENO-CU6 scheme

The WENO nonlinear weights of WENO-CU6 scheme are

$$\alpha_k = d_k \left(C + \frac{\tau_6}{\beta_k + \varepsilon} \right), \quad \omega_k = \frac{\alpha_k}{\sum_{k=0}^{3} \alpha_k},$$

- $\varepsilon = 10^{-40}$. (A very small number).
- Large constant $C \gg 1$ increases the contribution of the optimal weights. (Usually, $C = 20$ or larger).
- The global smoothness indicator

$$\tau_6 = \left| \beta_6 - \frac{1}{6}(\beta_0 + 4\beta_1 + \beta_2) \right| + O(\Delta x^6), \tag{3}$$

with a long and complex expression of β_6.

The WENO-η scheme

The WENO nonlinear weights of WENO-η are

$$\alpha_k = d_k \left(1 + \frac{\tau}{\eta_k + \varepsilon}\right), \quad \omega_k = \frac{\alpha_k}{\sum_{k=0}^{2} \alpha_k}. \tag{4}$$

- The local smoothness indicators

$$\eta_k = \sum_{m=1}^{r-1} [\Delta x^m P_{i-r+1+k}^{(m)}(x_i)]^2, \ k = 0, 1, 2. \tag{5}$$

  where $P_i^{(m)}(x)$ is the m_th derivative of the Lagrangian interpolation polynomial for approximating the value of the function $f(x)$ based on the values $(f_{i-r+1+k}, \dots, f_{i-r+1+k})$.
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The WENO-η scheme

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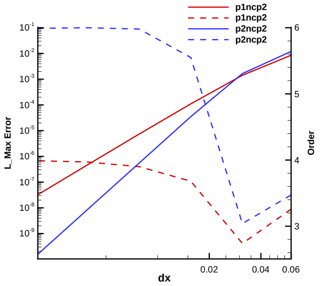
$$\tau_5 = |\eta_0 - \eta_2| + O(\Delta x^6). \quad (6)$$

$$\tau_6 = |6\eta_5 - (4\eta_1 + \eta_0 + \eta_2)| /6 + O(\Delta x^6). \quad (7)$$

$$\tau_8 = \left|(|P_0^{(1)}| - |P_2^{(1)}|)(P_0^{(2)} + P_2^{(2)} - 2P_1^{(2)})\right| + O(\Delta x^8). \quad (8)$$

WENO-$\eta(\tau_8)$, $n_{cp} = 2, \varepsilon = 10^{-40}$



The WENO-D Scheme

The improved WENO-Z scheme, which can guarantee the optimal order of accuracy in the presence of critical points, the nonlinear weights are

$$\alpha_k = d_k\left(1 + \Phi\left(\frac{\tau_{2r-1}}{\beta_k + \varepsilon}\right)^p\right), \quad \omega_k = \frac{\alpha_k}{\sum_{j=0}^{r-1}\alpha_j}, \quad k = 0, \ldots, r-1.$$

$$\Phi = \min\{1, \phi\}, \qquad \phi = \sqrt{|\beta_0 - 2\beta_1 + \beta_2|}. \tag{9}$$

Remark

- *Φ, as a linear combination of the β_k, can be treat as a shock sensor.*
    - *when the solution is smooth, $\Phi = \phi$.*
    - *around the shock, $\Phi = 1$, the new weights become the WENO-Z scheme.*
- *It can also be derived in other form, for example, a weighted linear combination of function values, says, $\{f_{i-1}, f_i, f_{i+1}\}$ (See WENO-η).*
- *The key point is that the following condition must be satisfied, namely,*

$$\phi\left(\frac{\tau_5}{\beta_k + \varepsilon}\right)^p \sim O(\Delta x^{r-1}). \tag{10}$$

to guarantee the formal order of accuracy regardless critical points.

In order to guarantee that the formal order of accuracy can be achieved regardless of critical points, the following condition must be satisfied, namely,

$$\phi\left(\frac{\tau_5}{\beta_k+\varepsilon}\right)^p \sim O(\Delta x^{r-1}). \tag{11}$$

By the Taylor expansions, one has ϕ^2 and τ_5 at x_i as

$$\phi^2 = \Big| \left(a_{13} f_i^{(1)} f_i^{(3)}\right) \Delta x^4 + \left(a_{15} f_i^{(1)} f_i^{(5)} + a_{24} f_i^{(2)} f_i^{(4)} + a_{33} (f_i^{(3)})^2\right) \Delta x^6 + \left(a_{17} f_i^{(1)} f_i^{(7)} + a_{26} f_i^{(2)} f_i^{(6)} + a_{35} f_i^{(3)} f_i^{(5)} + a_{44} (f_i^{(4)})^2\right) \Delta x^8 + O(\Delta x^9)\Big|. \tag{12}$$

$$\tau_5 = \Big| \left(a_{14} f_i^{(1)} f_i^{(4)} + a_{23} f_i^{(2)} f_i^{(3)}\right) \Delta x^5 + \left(a_{16} f_i^{(1)} f_i^{(6)} + a_{25} f_i^{(2)} f_i^{(5)} + a_{34} f_i^{(3)} f_i^{(4)}\right) \Delta x^7 + O(\Delta x^9)\Big|. \tag{13}$$

Definition of Order θ

Definition

The notation $\theta\left(g(\Delta x)\right)$ denotes the power of Δx in the leading term of the Taylor series expansion of $g(\Delta x)$, that is,

$$\theta\left(g\right) = n \iff g(\Delta x) = \Theta(\Delta x^n).$$

For instance, if $g(\Delta x) = 5\Delta x^2 + \Delta x^3$, then $\theta\left(g\right) = 2$.

According to Don et al. [2], the nonlinear component of the Z-type weights must satisfy

$$\phi\left(\frac{\tau_{2r-1}}{\beta_k+\varepsilon}\right)^p \geq \Delta x^{r-1}, \tag{14}$$

to guarantee the formal order of accuracy regardless of critical points, or

$$\theta\left(\phi\left(\frac{\tau_{2r-1}}{\beta_k+\varepsilon}\right)^p\right) \geq r-1. \tag{15}$$

$$\theta(\phi) + p\theta(\tau_{2r-1}) - p\theta(\beta_k+\varepsilon) \geq r-1, \tag{16}$$

$$\theta(\beta_k+\varepsilon) \leq \theta(\tau_{2r-1}) + \frac{\theta(\phi)-(r-1)}{p}, \tag{17}$$

$$\min\{\theta(\beta_k), \theta(\varepsilon)\} \leq \theta(\tau_{2r-1}) + \frac{\theta(\phi)-(r-1)}{p}. \tag{18}$$

[2] Don and Borges, J. Comput. Phys. 250, 2013

Analysis of Order of Accuracy

- If $\theta(\varepsilon) > \theta(\beta_k)$ implies $\varepsilon < \beta_k$, the non-linear component is dominated by β_k, the accuracy of the scheme won't be affected by ε.
- If $\theta(\varepsilon) \leq \theta(\beta_k)$, then equation (18) becomes

$$\theta(\varepsilon) \leq \theta(\tau_{2r-1}) + \frac{\theta(\phi) - (r-1)}{p}. \tag{19}$$

The integer parts of the optimal sensitivity order $\theta(\varepsilon)$, $\mathbf{m}$, for the fifth order WENO scheme$n_{cp} = 1, 2, 3$ and $p = 1, 2, 3$ are given in the table.

			WENO-Z(m)			WENO-D(m)		
n_{cp}	$\theta(\tau_5)$	$\theta(\phi)$	$p=1$	$p=2$	$p=3$	$p=1$	$p=2$	$p=3$
1	5	3	3	4	4	6	5	5
2	7	3	5	6	6	8	7	7
3	9	4	7	8	8	11	10	9

Based on the WENO-D scheme, a small modification on the non-linear weights is applied to the WENO-D scheme, namely,

$$\alpha_k = d_k \left(\max \left(1, \Phi \left(\frac{\tau_{2r-1}}{\beta_k + \varepsilon} \right)^p \right) \right). \tag{20}$$

This modification do not affect the optimal order at the presence of critical points as analysed above for the WENO-D scheme.

We name this improved WENO-D scheme as WENO-A scheme with A stands for Abarbanel.

Analysis of Order of Accuracy

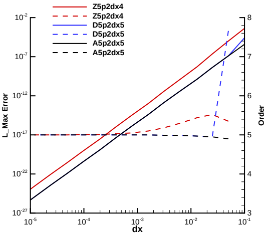
We will examine the performance of the WENO-Z, WENO-D and WENO-A scheme in achieving the formal order of accuracy for a smooth function in the presence of critical points.
Consider the following test function

$$f(x) = x^k e^{0.75x}, \quad x \in [-1, 1], \tag{21}$$

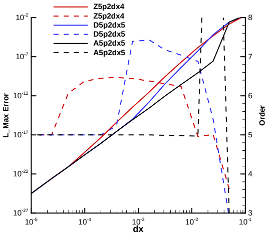
in which its first $k-1$ derivatives $f^{(j)}(0) = 0, j = 0, 1, ..., k-1$. That is, this function has a critical point of order $n_{cp} = k-1$ at $x = 0$.

Optimal Variable ε

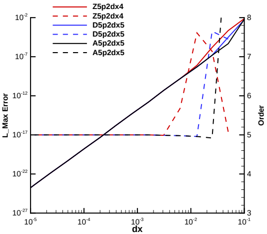
$n_{cp} = 1$



$n_{cp} = 2$

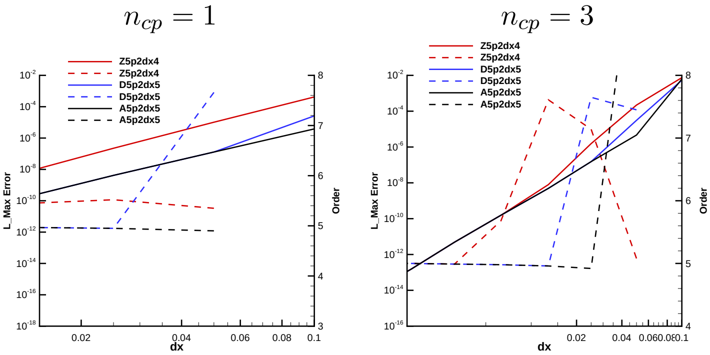


$n_{cp} = 3$



- Notice that for various $\varepsilon = \Delta x^m$ ($m = 4$ for WENO-Z scheme, $m = 5$ for WENO-D and WENO-A scheme)
  - All three methods achieve the optimal order asymptotically.
  - WENO-A and WENO-D have a smaller L_∞ error than WENO-Z.
  - For coarse mesh, WENO-A has smaller L_∞ error than WENO-D, and
  - WENO-A has a convergence quicker than WENO-D and WENO-Z schemes.

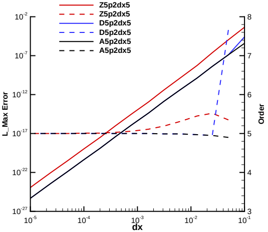
Analysis of Order of Accuracy
Optimal Variable ε



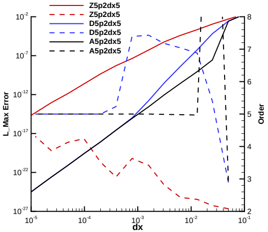
- The Zoomed in figure.
  - For coarse mesh, WENO-A has a smaller L_∞ error than WENO-D.

Analysis of Order of Accuracy
Fixed Variable ε

$n_{cp} = 1$



$n_{cp} = 2$



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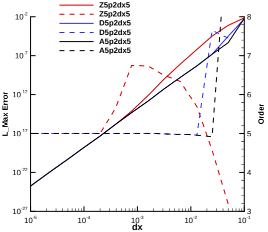
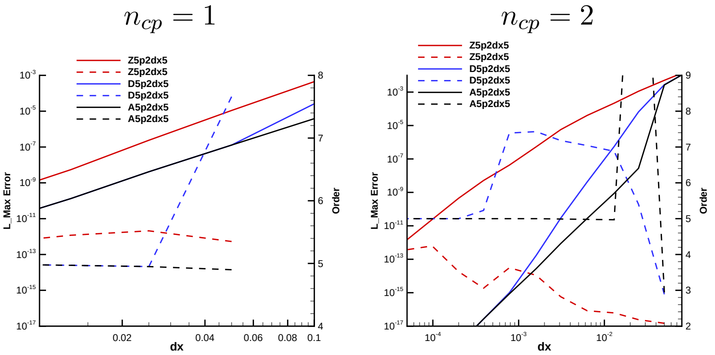


- Notice that for $\varepsilon = \Delta x^5$
  - Not all three methods can get optimal order.
  - WENO-A and WENO-D have smaller L_∞ error than WENO-Z.
  - WENO-A has a better convergence rate than the other two schemes.

Analysis of Order of Accuracy
Fixed Variable ε



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  - For coarse mesh, WENO-A has smaller L_∞ error than WENO-D.

WENO-A Scheme

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We compare the numerical performance of WENO-A, WENO-D and classical WENO-Z scheme. For those numerical examples, the flow is describes by the Euler equations

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ (E+p)u \end{pmatrix}_x + \begin{pmatrix} \rho v \\ \rho vu \\ \rho v^2 + p \\ \rho vw \\ (E+p)v \end{pmatrix}_y + \begin{pmatrix} \rho w \\ \rho wu \\ \rho wv \\ \rho w^2 + p \\ (E+p)w \end{pmatrix}_z = 0.$$

This set of equations describes the conservation laws expressed by mass density ρ, momentum density $\rho\mathbf{v} \equiv (\rho u, \rho v, \rho w)$ and total energy density $E = \rho e + \frac{1}{2}\rho\mathbf{v}^2$, where e is the internal energy per unit mass. To close this set of equations, the ideal-gas equation of state $p = (\gamma - 1)\rho e$ with $\gamma = 1.4$ is used.

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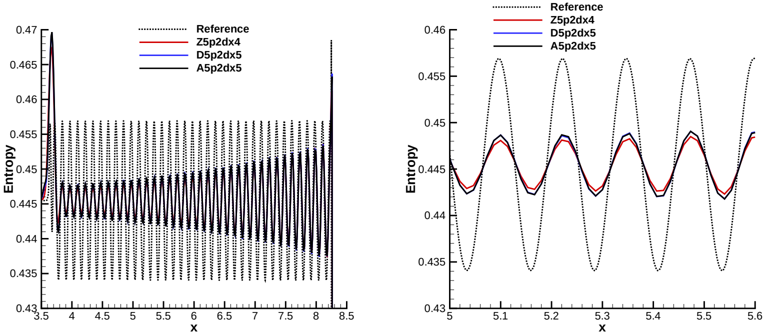
Numerical Result

For those numerical experiments, the Euler equations are solved by following the general WENO methodology.

- Characteristic projection by Roe averaged eigensystem.
- Flux splitting by local Lax-Friedrichs.
- Time integration by third order TVD Runge-Kutta method with CFL number CFL$= 0.45$.

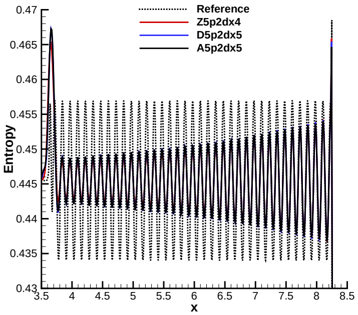
One Dimension Shock Entropy Problem

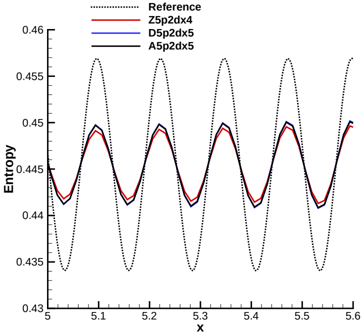
The final time is $t = 5$ and resolution is $N = 1500$.



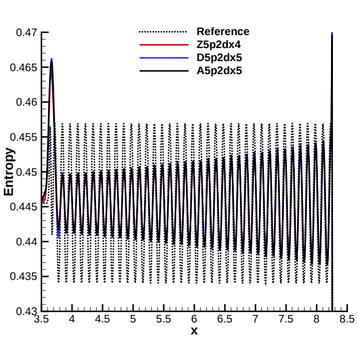
One Dimension Shock Entropy Problem

The final time is $t = 5$ and resolution is $N = 1600$.

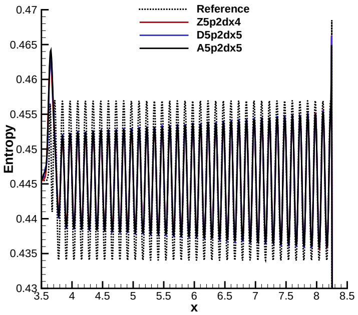


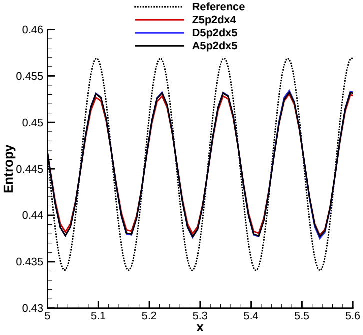


The final time is $t = 5$ and resolution is $N = 1700$.



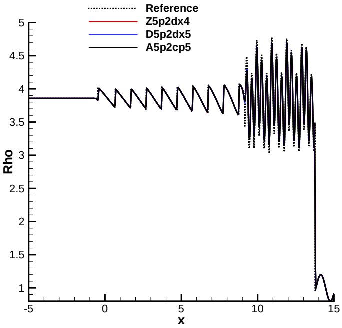
The final time is $t = 5$ and resolution is $N = 2000$.

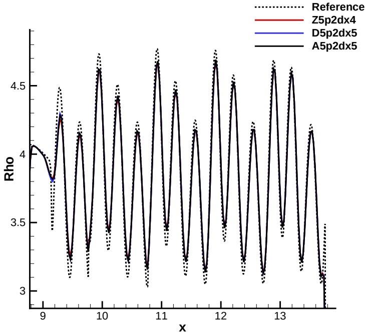




One Dimension Shock Density Problem

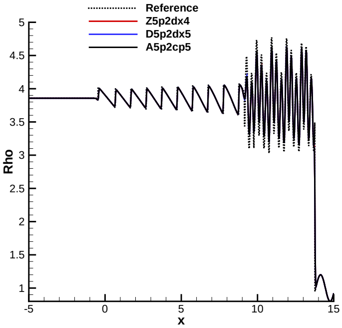
The final time is $t = 5$ and resolution is $N = 800$.

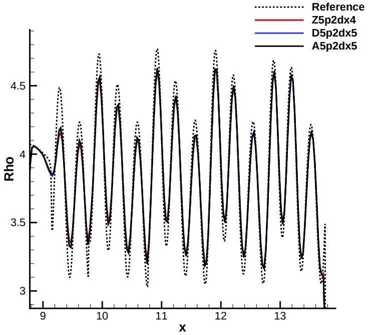




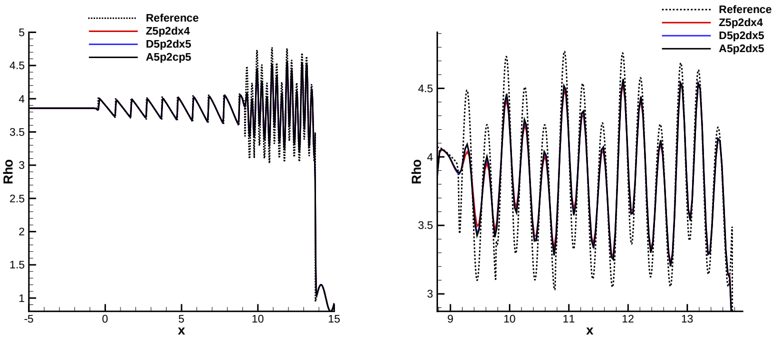
One Dimension Shock Density Problem

The final time is $t = 5$ and resolution is $N = 700$.



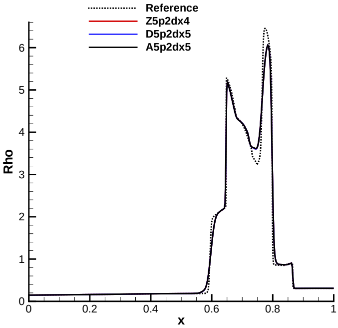


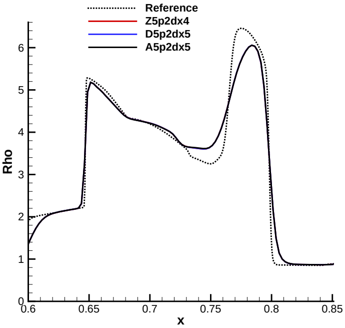
The final time is $t = 5$ and resolution is $N = 600$.



Interacting Blast Wave Problem

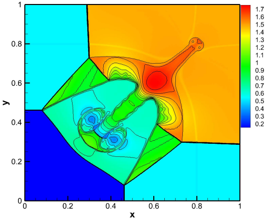
The final time is $t = 0.038$ and resolution is $N = 400$.



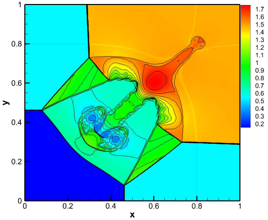


The final time is $t = 0.8$ and resolution is $N = 400 \times 400$.

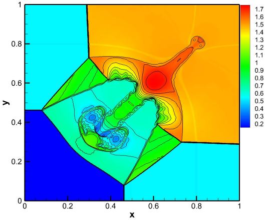
WENO-Z5



WENO-D5

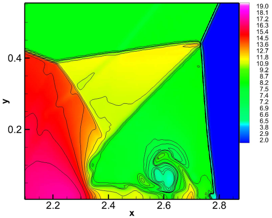


WENO-A5

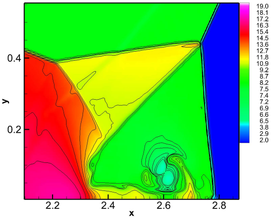


The final time is $t = 0.2$ and resolution is $N = 800 \times 200$.

WENO-Z5



WENO-D5



WENO-A5

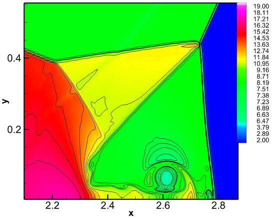
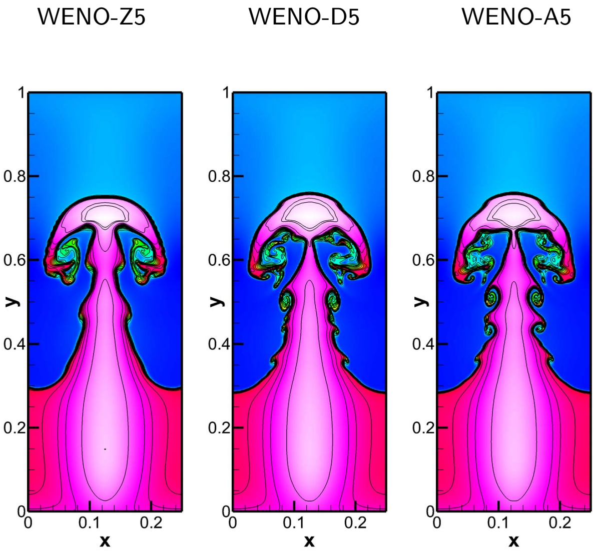


Table: The CPU time (in seconds) of the DMR problem.

$N \times M$	WENO-Z	WENO-D	WENO-A
800×200	4.1E+03	4.6E+03	4.5E+03

Rayleigh-Taylor Instability Problem



$N = \times$ higher resolution will be updated in the next version, the code is still running

Use WENO-A scheme for solving overestimated quasi-conservative form of the compressible multicomponent flows simulation.

$$\frac{\partial \mathbf{Q}}{\partial t} + \mathbf{U}\frac{\partial \mathbf{F}}{\partial x} = 0, \tag{22}$$

$$\mathbf{Q} = \begin{bmatrix} \rho \\ \rho u \\ \rho e \\ \rho Y_1 \\ \gamma_p \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & u \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + P \\ u\left(\rho e + P\right) \\ \rho u Y_1 \\ \gamma_p \end{bmatrix}. \tag{23}$$

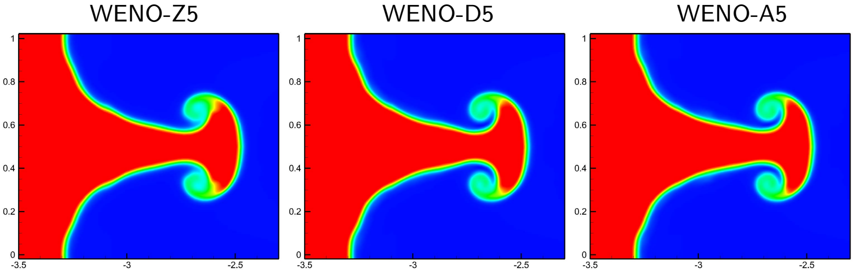
The initial conditions are

$$(\rho \ u \ v \ P \ \gamma \ M) = \begin{cases} (\ 1.4112 & -0.3613 & 0 & 1.6272/1.4 & 1.4 & 28.8 \ ), & x > -0.8 \\ (\ 5.04 & 0 & 0 & 1/1.4 & 1.093 & 145.15 \ ), & x < x_0 \\ (\ 1 & 0 & 0 & 1/1.4 & 1.4 & 28.8 \ ), & \text{otherwise.} \end{cases} \tag{24}$$

- where $x_0 = -1.1 - 0.1\cos(2\pi y)$.
- The computational domain is $-8 \leq x \leq 0$ and $0 \leq y \leq 1$.

Richtmyer-Meshkov Instability Problem

The final time is $t = 8.25$ and resolution is $N = 1024 \times 128$.



Shock-Bubble Interaction Problem

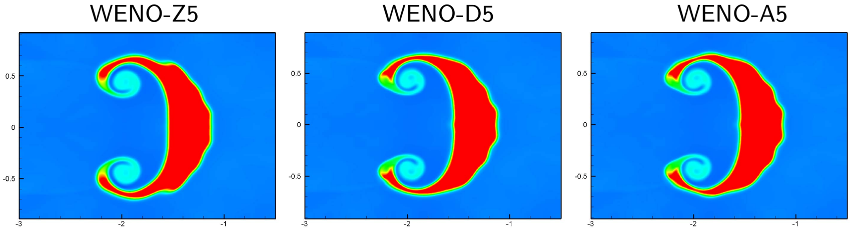
The initial conditions are

$$(\rho\ u\ v\ P\ \gamma\ M) = \begin{cases} (\ 1.3764 & -0.3336 & 0 & 1.5698/1.4 & 1.4 & 28.80\ ), & x \geq 1.0 \\ (\ 3.153 & 0 & 0 & 1/1.4 & 1.249 & 90.82\ ), & \sqrt{x^2+y^2} < 0.5 \\ (\ 1 & 0 & 0 & 1/1.4 & 1.4 & 28.80\ ), & \text{otherwise.} \end{cases} \quad (25)$$

- The computational domain is $-3.5 \leq x \leq 3$ and $-1 \leq y \leq 1$.

Shock-Bubble Interaction Problem

The final time is $t = 7.337$ and resolution is $N = 650 \times 178$.



Conclusion And Future Work

- **Summary and conclusion remark**
  - We present the WENO-D and WENO-A schemes for the solution of nonlinear hyperbolic conservation laws.
  - We analyzed the new schemes in resolving function with a high order critical point.
  - We demonstrate that the new schemes can achieve the optimal order of accuracy with a greatly relaxed constraint on the sensitivity parameter ε.
  - The WENO-A scheme has a substantially smaller ε than the standard WENO-Z scheme, and performs competitively for shocked flows.
- **Future Work**
  - Extend the WENO-A scheme to higher order.
  - Extend the WENO-A scheme to alternative WENO scheme (AWENO) scheme for multi-components shocked flows in a general curvilinear coordination.

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