A source of uncertainty in computed discontinuous flows

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$$\sum_{i=0}^{m} \left(\psi_{i,z_j}(z)\right)_{x_i} = 0, \qquad j = 1, \cdots, n$$

$\mathrm{x} \in \Omega \in \mathbb{R}^m$, $\nu(x)$ a.e. on $\partial\Omega$ $z(x) \in D \subseteq \mathbb{R}^n$

Solution Set $\mathcal{S}$

Regularity $z \in C(\overline{\Omega}/\Gamma)^n$

Admissibility $\nabla \cdot q(z) \leq 0, \quad q_i(z) = z \cdot \psi^{\dagger}_{i,z}(z) - \psi_i(z), \mathrm{i} = 1, \cdots, m$

Recoverability $\mathcal{D} = \mathcal{B}(\mathcal{S})$

$\mathrm{b} = \mathcal{B}(z) = (I - P)\psi^{\dagger}_{\nu,z}(z)$

$\mathcal{D} \subseteq \ker P \; \mathcal{A} = \mathcal{B}^{-1} \in C(\mathcal{D} \to \mathcal{S})$?

Approximation scheme

$$\mathcal{A}_\delta \in C(\mathcal{D} \to \mathcal{G}), \qquad \delta > 0, \qquad \mathcal{S} \subset \mathcal{G}$$

$$z_\delta(b) = \mathcal{A}_\delta(b) \in \mathcal{G}: \iint_\Omega \sum_{i=1}^{m} \sum_{j=1}^{n} \psi_{i,z_j}(z_\delta)\theta_{j,x_i} \xrightarrow{\delta \downarrow 0} \int_{\partial\Omega} b \cdot \theta$$

$$for \text{ all } \theta \in X = \{\phi \in \left(C^1(\Omega) \cap C(\overline{\Omega})\right)^n, P\phi \Big|_{\partial\Omega} = 0\}$$

$Suppose$ $\quad z_\delta(b) \xrightarrow{\delta \downarrow 0} z_0(b)\ strongly\ in\ \mathcal{G}$

$z_0(b) \in \mathcal{S}, \qquad \mathcal{B}(z_0) = b$

$z_0(b) = \mathcal{A}(b)$?

$\mathcal{B} \circ \mathcal{A} = I\ on\ \mathcal{D}$

$\mathcal{A} \circ \mathcal{B} = I\ on\ \mathcal{S}$?

Linear case

$$\sum_{i=1}^{m} V_i z_{x_i} = 0 \;\; \psi_i(z) = q_i(z) = \tfrac{1}{2} z \cdot V_i z, i = 1, \cdots, m$$

$$b = (I - P)V_\nu z$$

$$Conservation\ of\ Energy\ \nabla \cdot q(z) = 0$$

$$\mathrm{q(z)} = \nabla\Phi(z) + \Psi(z)$$

$$\Delta\Phi = 0\ in\ \Omega \qquad \nabla \cdot \Psi = 0\ in\ \Omega$$

$$\nu \cdot \nabla\Phi = q_\nu\ on\ \partial\Omega \qquad \Psi\Big|_{\partial\Omega} = 0$$

Theorem: *Assume* P *such that* $PV_\nu P \geq 0$ *a.e. on* $\partial\Omega$
Then b *uniquely determines* $V_\nu z$ *and* Φ
(up to an additive constant) but not Ψ

$Assume\ \mathcal{A}\ Frechet\ differentiable$

$$d\mathcal{A}(z){:}\,\mathcal{D}' \to \mathcal{S}'(z)$$

$$\int_{\partial\Omega} b \cdot \theta = \iint_{\Omega} \sum_{i=1}^{m} \sum_{j=1}^{n} \psi_{i,z_j}(z)\theta_{j,x_i}$$

$$\int_{\partial\Omega} \dot{b} \cdot \theta = \iint_{\Omega\setminus\Gamma} \sum_{i=1}^{m} \sum_{j=1}^{n} \psi_{i,z_j z_k}(z)\dot{z}_k\theta_{j,x_i} - \int_{\Gamma} (\hat{\mu} \cdot \dot{x}) \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\psi_{i,z_j}(z)\right] \theta_{j,x_i}$$

$$= \iint_{\Omega/\Gamma} \dot{z} \cdot R\theta + \int_{\Gamma} \sigma(S\theta)$$

$$\dot{b} \in \mathcal{D}', \qquad \sigma = -\hat{\mu} \cdot \dot{x}\ a.e.\ on\ \Gamma, \qquad (\dot{z}, \sigma) \in \mathcal{S}'(z)$$

$$(R\theta)_k = \sum_{i=1}^{m} \sum_{j=1}^{n} \psi_{i,z_j z_k}(z)\theta_{j,x_i}, k = 1, \dots, n$$

$$(S\theta) = \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\psi_{i,z_j}(z)\right] \theta_{j,x_i}$$

$Existence\ and\ boundedness\ of\ (\dot{z},\sigma)$

$$\|\theta\|_{\partial\Omega} = \operatorname{lub}_{\dot{b}} \frac{\int_{\partial\Omega} \dot{b}\cdot\theta}{\|\dot{b}\|_{\mathcal{D}'}} \qquad \|\theta\| = \operatorname{lub}_{(\dot{z},\sigma)} \frac{\iint_{\Omega\backslash\Gamma} \dot{z}\cdot R\theta + \int_{\Gamma} \sigma(S\theta)}{\|(\dot{z},\sigma)\|_{\mathcal{S}'(z)}}$$

$$\|(\dot{z},\sigma)\|_{\mathcal{S}'(z)} \le c_z \|\dot{b}\|_{\mathcal{D}'} \Leftrightarrow \|\theta\|_{\partial\Omega} \le c_z \|\theta\|$$

requires ker P sufficiently small, and
$ker\ \psi_{\nu,zz}(z) \subseteq range\ P$

$Z\ completion\ of\ X\ in\ norm\ \|.\|\ (depends\ on\ z \in \mathcal{S})$

$Theorem$: $Assume\ estimate\ (for\ \|\theta\|_{\partial\Omega}), that\ \mathcal{D}' \subseteq (Z|_{\partial\Omega})^*, and\ that$ $\|\theta\|$ *is Frechet differentiable.*

$Then\ for\ any\ \dot{b} \in \mathcal{D}', there\ exist\ (\dot{z},\sigma) \in (RZ)^* \times (SZ)^*\ satisfying$

$$\int_{\partial\Omega} \dot{b}\cdot\theta = \iint_{\Omega\backslash\Gamma} \dot{z}\cdot R\theta + \int_{\Gamma} \sigma(S\theta) \quad for\ all\ \theta \in Z$$

$$Proof: \ for\ \xi, \theta \in Z, \qquad d(\|\xi\|)\theta = \iint\limits_{\Omega/\Gamma} \zeta(\xi)\cdot R\theta + \int\limits_{\Gamma} \zeta_{\Gamma}(\xi)(S\theta)$$

$$For\ all\ \theta \in Z, J_{\dot{b}}(c) = -\int\limits_{2\Omega} \dot{b}\cdot\theta + \tfrac{1}{2}\|\theta\|^2 \geq -\frac{c_Z^2}{2}\left\|\dot{b}\right\|_{\mathcal{D}'}^2,$$

$At\ any\ stationary\ point\ \xi \in Z\ of\ J_{\dot{b}},$

$$\int\limits_{\partial\Omega} \dot{b}\cdot\theta = \|\xi\| d(\|\xi\|)\theta \ \ for\ all\ \theta \in Z$$

$$\dot{z} = \|\xi\|\zeta(\xi), \qquad \sigma = \|\xi\|\zeta_{\Gamma}(\xi)$$

$$Uniqueness\ of\ (\dot{z},\sigma) \in \mathcal{S}'(z)$$

$$Identified\ with\ \mathcal{A} \circ \mathcal{B} = I\ on\ \mathcal{S}$$

$$Uniqueness\ failure, nontrivial\ (f,g) \in \mathcal{S}'(z), f \in (RZ)^*, g \in (SZ)^*$$

$$\iint_{\Omega/\Gamma} f \cdot R\theta + \int_{\Gamma} g(S\theta) = 0 \quad for\ all\ \theta \in Z$$

$$Necessarily\ R^{\dagger} f = 0\ in\ \Omega\backslash\Gamma, \qquad (I-P)\psi_{\nu,zz}(z) f = 0\ \ a.e.on\ \partial\Omega$$

$$\psi_{\nu,zz}(z) f \,\Big|_{\partial\Omega} \subset range\ P$$

$$\Lambda = span\left\{(\dot{z},\sigma), (\dot{b} \in \mathcal{D}')\right\}$$

$$\Lambda \subseteq \mathcal{S}'(z) \subseteq (RZ)^* \times (SZ)^*$$

$$Uniqueness\ of\ (\dot{z},\sigma)\ in\ (RZ)^* \times (SZ)^*\ implies$$

$$Z = Z_{\Omega\backslash\Gamma} \bigoplus Z_{\Gamma}$$

$$SZ_{\Omega\backslash\Gamma} = 0\ \ on\ \Gamma$$

$$RZ_{\Gamma} = 0\ in\ \Omega/\Gamma$$

$Z = Z_{\Omega\backslash\Gamma} \oplus Z_\Gamma$ *requires* $\ker P$ *sufficiently large*

Z_0 *completion* $(C^1(\Omega) \cap C(\overline{\Omega}))^n$, *no boundary conditions*

Assume $Z_0 = Z_{0,\Omega\backslash\Gamma} \oplus Z_{0,\Gamma}$, *denote* $Z_{0,\cap} = Z_{0,\Omega\backslash\Gamma} \cap Z_{0,\Gamma}$

$$RZ_{0,\cap} = SZ_{0,\cap} = 0$$

Theorem: *Assume* P *such that* $\{PZ_{0,\Gamma}\} = \{PZ_{0,\cap}\}$

Then $Z = Z_{\Omega/\Gamma} \oplus Z_\Gamma$

Investigation

Equivalent norms $\|\theta\|_\lambda, \qquad \lambda = 1{,}2$

Same $Z, (RZ)^*, (SZ)^*, \Lambda, \mathcal{S}'$

But $d(\|\theta\|_1) \neq d(\|\theta\|_2), \xi_1 \neq \xi_2$ *minimizing* $-\int_{\partial\Omega} \dot{b}\cdot\theta + \frac{1}{2}\|\theta\|_\lambda^2$

will get $(\dot{z}_1,\sigma_1)$ *and* $(\dot{z}_2,\sigma_2)$

If $(\dot{z},\sigma)$ *is unique within* $\Lambda, (\dot{z}_1,\sigma_1) = (\dot{z}_2,\sigma_2)$

If $(\dot{z}_1,\sigma_1) \neq (\dot{z}_2,\sigma_2) \qquad f = \dot{z}_1 - \dot{z}_2 \; g = \sigma_1 - \sigma_2$

Example: $\|\theta\|_\lambda^2 = \iint_{\Omega\backslash\Gamma} w|R\theta|^2 + \lambda\int_\Gamma w_\Gamma (S\theta)^2 \quad w(x), w_\Gamma(x) \geq 1$

$$\|(\dot{z},\sigma)\|_{\mathcal{S}'}^2 = \iint_{\Omega\backslash\Gamma} \frac{1}{w}|\dot{z}|^2 + \int_\Gamma \frac{1}{w_\Gamma}\sigma^2 \qquad (RZ)^* = wRZ, (SZ)^* = w_\Gamma SZ$$

$$\dot{z}_\lambda = wR\xi_\lambda \,, \sigma_\lambda = \lambda w_\Gamma S\xi_\lambda$$

$Theorem\colon Suppose\ \dot{z}_1 = \dot{z}_2\ or\ \sigma_1 = \sigma_2\ for\ each\ \dot{b} \in \mathcal{D}'$

$Then\ both\ hold\ and\ Z_{\Omega\backslash\Gamma}, Z_{\Gamma}\ are\ sufficiently\ large\ that$

$\Lambda \subseteq wRZ_{\Omega\backslash\Gamma} \times w_{\Gamma}SZ_{\Gamma}$

$\dot{z} \in wRZ_{\Omega\backslash\Gamma}, \sigma \in w_{\Gamma}SZ_{\Gamma}\ \ for\ any\ \ \dot{b} \in \mathcal{D}'$

Computation

$Convergence\ z_\delta \to z_0\ suggests\ boundary\ data\ not\ over-specified,$
$\ker P\ sufficiently\ small\ \ \|\theta\|_{\partial\Omega} \le c_z\|\theta\|, for\ suitable\ w, w_\Gamma, \|\cdot\|_{\partial\Omega}$

$Find\ \ w, w_\Gamma, \|\cdot\|_{\partial\Omega}\ such\ that\ J(Z) \ge -\tfrac{1}{2}c_z^2, J(\theta) = -\|\theta\|_{\partial\Omega} + \tfrac{1}{2}\|\theta\|^2\ ,$

$\|\theta\|_{\partial\Omega}\ determines\ \|\dot{b}\|_{\mathcal{D}'}\ , for\ example \|\theta\|_{\partial\Omega} = \|Q\theta\|_{L_2(\partial\Omega)}$
$implies \|\dot{b}\|_{\mathcal{D}'} = \|Q^{-1}\dot{b}\|_{L_2(\partial\Omega)}$
$Can\ be\ easy, w = w_\Gamma = 1\ , Q = I, depending\ on\ z_0$

$$Then\ for\ any\ \dot{b} \in \mathcal{D}', J_{\dot{b}\lambda}(Z) \ge -c_{\dot{b}}\ ,\ J_{\dot{b}\lambda}(\theta) = -\int_{\partial\Omega} \dot{b}\cdot\theta + \tfrac{1}{2}\|\theta\|_\lambda^2$$

$\xi_{\dot{b}\lambda}\ minimizing\ J_{\dot{b}\lambda}\ over\ Z\ satisfies$

$$\int_{2\Omega} \dot{b}\cdot\theta = \iint_{\Omega\setminus\Gamma} wR\xi_{\dot{b}\lambda}\cdot R\theta + \lambda\int_{\Gamma} w_\Gamma(S\xi_{\dot{b}\lambda})(S\theta)$$

$for\ all\ \theta \in Z.\ Suffices\ to\ determine\ if$

$$R(\xi_{\dot{b}1} - \xi_{\dot{b}2}) = 0$$

$Suffices\ to\ discretize\ four\ terms$

$$\int_{2\Omega} \dot{b}\cdot\theta\ , \qquad \|\theta\|_{\partial\Omega}\ , \qquad \iint_{\Omega\setminus\Gamma} w|R\theta|^2\ , \qquad \int_{\Gamma} w_\Gamma (S\theta)^2$$

$Have\ z_\delta \xrightarrow{\delta\downarrow 0} z_0$

$$(R_\delta\theta)_k = \sum_{i=1}^{m}\sum_{j=1}^{n} \psi_{i,z_j z_k}(z_\delta)\theta_{j,x_i} \quad , k = 1,\cdots,n\ , throughout\ \Omega$$

$Need\ w_\delta \xrightarrow{\delta\downarrow 0} w\ so\ that$

$$\left\| w_\delta^{1/2} R_\delta\theta - w^{1/2} R\ \theta \right\|_{L_2(\Omega)} \xrightarrow{\delta\downarrow 0} 0\ for\ any\ \theta \in \left(W^{1,\infty}(\Omega)\right)^n$$

$Don't\ have\ \Gamma\ ; \left[\psi_{i,z_j}(z_\delta)\right] undefined$

$Have$ "transition regions" Γ_ε of size $O(\varepsilon)$

$\varepsilon = o(1)\ \ as\ \ \delta \downarrow 0$

z_δ "well behaved" $outside\ of\ \Gamma_\varepsilon$

$$Crude\ approximation\ of\ S\theta\ suffices$$

$$S_{\varepsilon\delta} = S_\varepsilon(z_\delta) \in C\left(\left(W^{1,\infty}(\Omega)\right)^n \to \left(L_\infty(\Omega)\right)^m\right)$$

$$S_{\varepsilon\delta}(\theta)_l(x) \approx \varepsilon^{-1/2} \sum_{i=1}^{m} \sum_{j=1}^{n} \left(\psi_{i,z_j}\left(z_\delta\left(x + \frac{\varepsilon}{4}\hat{x}_l\right)\right) - \psi_{i,z_j}\left(z_\delta\left(x - \frac{\varepsilon}{4}\hat{x}_l\right)\right)\right)\theta_{j,x_i}(x)$$

$$throughout\ \Omega \qquad l = 1, \ldots, m$$

$Need\ \delta\ sufficiently\ small, depending\ on\ \varepsilon, and\ w_{\Gamma_\varepsilon}$

$such\ that\ for\ any\ \theta \in \left(W^{1,\infty}(\Omega)\right)^n$

$$\iint_{\Omega} w_{\Gamma_\varepsilon} |S_{\varepsilon\delta}\theta - S_{\varepsilon 0}\theta|^2 \xrightarrow{\varepsilon,\delta\downarrow 0} 0$$

and

$$\iint_{\Omega\setminus\Gamma_\varepsilon} w_{\Gamma_\varepsilon} |S_{\varepsilon\delta}\theta|^2 \xrightarrow{\varepsilon,\delta\downarrow 0} 0$$

Then approximate $J, J_{\dot{b}\cdot\lambda}$ *by*

$$J_{\varepsilon\delta}(\theta) = -\|\theta\|_{\partial\Omega} + \frac{1}{2}\int_{\Omega} w_{\delta}|R_{\delta}\theta|^2 + \frac{1}{2}\int_{\Omega} w_{\Gamma_{\varepsilon}}|S_{\varepsilon\delta}\theta|^2$$

$$J_{\dot{b}\varepsilon\delta\lambda}(\theta) = -\int_{\partial\Omega} \dot{b}\cdot\theta + \frac{1}{2}\int_{\Omega} w_{\delta}|R_{\delta}\theta|^2 + \frac{\lambda}{2}\int_{\Omega} w_{\Gamma_{\varepsilon}}|S_{\varepsilon\delta}\theta|^2 \;, \lambda = 1,2$$

for $\theta \in X_{\delta}$: *nested, finite* $-$ *dimensional subspace of* $\left(W^{1,\infty}(\Omega)\right)^n$,

becoming dense in X *as* $\delta \downarrow 0$, $PX_{\delta}\Big|_{\partial\Omega} = 0$

X_{δ} *does* **not** *have to be the test space associated with* z_{δ}.

$Seek\ w_\delta\ , w_{\Gamma_\varepsilon}\ , \|\cdot\|_{\partial\Omega}\ , X_\delta\ , \delta(\varepsilon)\ to\ "verify"$

$$(*)\ \underset{\varepsilon,\delta\downarrow 0}{glb}\ J_{\varepsilon\delta}(X_\delta) \geq -c$$

$Theorem: Assume\ (*)\ and\ w_{\Gamma_\varepsilon}\ , w_\Gamma\ (\geq 1)\ \ such\ that\ for\ any\ \theta \in X$

$$\limsup_{\varepsilon\downarrow 0} \iint_\Omega w_{\Gamma_\varepsilon} |S_{\varepsilon 0}\theta|^2 \leq c \int_\Gamma w_\Gamma (S\theta)^2$$

$Then$

$$(**) \qquad J(X) \geq -c$$

$Theorem: Assume\ (**)\ and\ w_{\Gamma_\varepsilon}\ , w_\Gamma\ such\ that\ for\ any\ \theta \in X$

$$\liminf_{\varepsilon\downarrow 0} \iint_\Omega w_{\Gamma_\varepsilon} |S_{\varepsilon 0}\theta|^2 \geq \frac{1}{c} \int_\Gamma w_\Gamma (S\theta)^2$$

$Then\ for\ any\ \theta \in X$

$$\liminf_{\varepsilon,\delta\downarrow 0} J_{\varepsilon\delta}(\theta) \geq -c$$

$with\ a\ constant\ indepenent\ of\ \theta, but\ don't\ get\ (*)$

$\|\cdot\|_{\partial\Omega}$ *determines* $\|\cdot\|_{\mathcal{D}'}$. *If* $(*)$ *holds, then for any* $\dot{b} \in \mathcal{D}'$

$J_{\dot{b}\varepsilon\delta\lambda}(X_\delta) \geq -c_{\dot{b}}$

$\xi_{\dot{b}\varepsilon\delta\lambda} \in X_\delta$ *minimizing* $J_{\dot{b}\varepsilon\delta\lambda}$ *over* X_δ *satisfies*

$$\int_{\partial\Omega} \dot{b}\cdot\theta_\delta = \iint_\Omega w_\delta R_\delta \xi_{\dot{b}\varepsilon\delta\lambda}\cdot R_\delta\theta_\delta + \lambda \iint_\Omega w_{\Gamma_\varepsilon} S_{\varepsilon\delta}\xi_{\dot{b}\varepsilon\delta\lambda}\cdot S_{\varepsilon\delta}\theta_\delta \ \textit{for all}\ \theta_\delta \in X_\delta$$

Theorem: *Assume that any sequence* $\phi_\delta \in X_\delta$ *satisfying* $\iint_\Omega w_{\Gamma_\varepsilon}|S_{\varepsilon\delta}\phi_\delta|^2 \leq c$

contains a subsequence with a (weak) limit $\phi_0 \in Z$ *such that for any* $\theta \in X$

$$\iint_\Omega w_{\Gamma_\varepsilon} S_{\varepsilon\delta}\phi_\delta \cdot S_{\varepsilon 0}\theta \xrightarrow{\varepsilon,\delta\downarrow 0} \int_\Gamma w_\Gamma (S\phi_0)(S\theta)$$

Then $w_\delta^{1/2} R_\delta(\xi_{\dot{b}\varepsilon\delta 1} - \xi_{\dot{b}\varepsilon\delta 2}) \overset{\varepsilon,\delta\downarrow 0}{\rightharpoonup} 0$ *weakly in* $\left(L_2(\Omega)\right)^n$

implies $R\xi_{\dot{b}1} = R\xi_{\dot{b}2}$

$(1-\Delta)^{-1} w_\delta^{1/2} R_\delta(\xi_{\dot{b}\varepsilon\delta 1} - \xi_{\dot{b}\varepsilon\delta 2}) \xrightarrow{\varepsilon,\delta\downarrow 0} 0$ *in* $\left(L_2(\Omega)\right)^n$ suffices