CONSERVATION LAWS ON THE SPHERE: FROM SHALLOW WATER TO BURGERS

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Institute of Mathematics, Hebrew University, Jerusalem, Israel

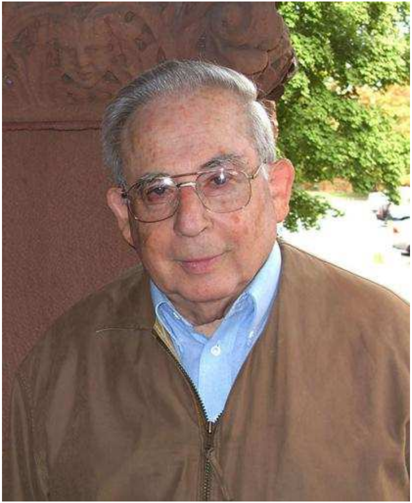
Advances in Applied Mathematics

IN MEMORIAM OF PROFESSOR SAUL ABARBANEL

Tel Aviv University

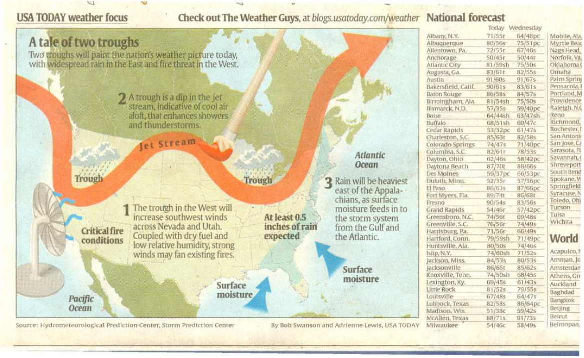
December 2018

joint work with JOSEPH FALCOVITZ, PHILIPPE LEFLOCH



"...together with David Gottlieb we noticed that some of the stuff that people were doing, the formulation was not strongly well posed, which is a mathematical point of view. So we got interested in how to make it more posed." (Interview with P. Davis, Brown University,2003).
"Problems should be studied in a 'physico-mathematical' fashion"–(private communication)

General Circulation Model –JETSTREAM



USA TODAY weather focus

Check out The Weather Guys, at *blogs.usatoday.com/weather*

A tale of two troughs

Two troughs will paint the nation's weather picture today, with widespread rain in the East and fire threat in the West.

2 A trough is a dip in the jet stream, indicative of cool air aloft, that enhances showers and thunderstorms.

Jet Stream

Trough

Trough

Atlantic Ocean

3 Rain will be heaviest east of the Appalachians, as surface moisture feeds in to the storm system from the Gulf and the Atlantic.

1 The trough in the West will increase southwest winds across Nevada and Utah. Coupled with dry fuel and low relative humidity, strong winds may fan existing fires.

Critical fire conditions

At least 0.5 inches of rain expected

Surface moisture

Surface moisture

Pacific Ocean

Source: Hydrometeorological Prediction Center, Storm Prediction Center

By Bob Swanson and Adrienne Lewis, USA TODAY

National forecast

	Today	Wednesday
Albany, N.Y.	71/55r	64/48pc
Albuquerque	80/56s	75/51pc
Allentown, Pa.	72/55r	67/46s
Anchorage	50/45r	50/44r
Atlantic City	81/59sh	75/50s
Augusta, Ga.	83/61t	82/55s
Austin	91/60s	91/67s
Bakersfield, Calif.	90/61s	83/61s
Baton Rouge	86/58s	84/57s
Birmingham, Ala.	81/54sh	75/50s
Bismarck, N.D.	57/35s	59/40pc
Boise	64/44sh	63/47sh
Buffalo	68/51sh	60/47c
Cedar Rapids	53/32pc	61/47s
Charleston, S.C.	85/63t	82/58s
Colorado Springs	74/47s	71/40pc
Columbia, S.C.	82/61r	78/53s
Dayton, Ohio	62/46s	58/42pc
Daytona Beach	87/70t	86/66s
Des Moines	59/37pc	66/53pc
Duluth, Minn.	52/35r	57/36pc
El Paso	86/63s	87/66pc
Fort Myers, Fla.	89/74t	86/68t
Fresno	90/54s	83/56s
Grand Rapids	54/46r	57/42pc
Greensboro, N.C.	74/56t	69/48s
Greenville, S.C.	76/56r	74/49s
Harrisburg, Pa.	71/56r	66/49s
Hartford, Conn.	79/59sh	71/49pc
Huntsville, Ala.	80/50s	74/46s
Islip, N.Y.	74/60sh	71/52s
Jackson, Miss.	84/53s	80/53s
Jacksonville	86/65t	85/62s
Knoxville, Tenn.	74/50sh	68/45s
Lexington, Ky.	69/45s	61/43s
Little Rock	81/52s	79/55s
Louisville	67/48s	64/47s
Lubbock, Texas	82/58s	86/64pc
Madison, Wis.	51/38c	59/42s
McAllen, Texas	88/71s	91/73s
Milwaukee	54/46c	58/49s

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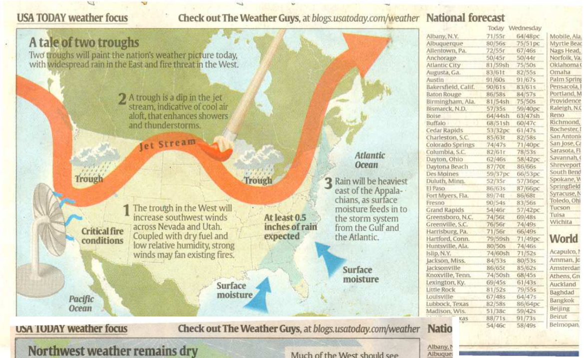
General Circulation Model –JETSTREAM



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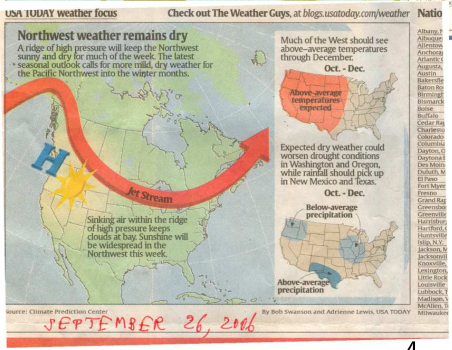

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Northwest weather remains dry

A ridge of high pressure will keep the Northwest sunny and dry for much of the week. The latest seasonal outlook calls for more mild, dry weather for the Pacific Northwest into the winter months.

H

Jet Stream

Sinking air within the ridge of high pressure keeps clouds at bay. Sunshine will be widespread in the Northwest this week.

Much of the West should see above–average temperatures through December.

Oct. - Dec.

Above-average temperatures expected

Expected dry weather could worsen drought conditions in Washington and Oregon, while rainfall should pick up in New Mexico and Texas.

Oct. - Dec.

Below-average precipitation

Above-average precipitation

Source: Climate Prediction Center

By Bob Swanson and Adrienne Lewis, USA TODAY

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Little Rock
Louisville
Lubbock, T
Madison, V
McAllen, T
Milwaukee

SEPTEMBER 26, 2006

MOVING VORTEX

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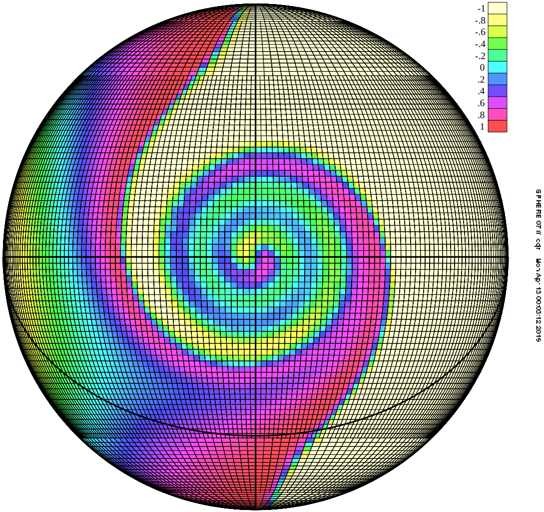

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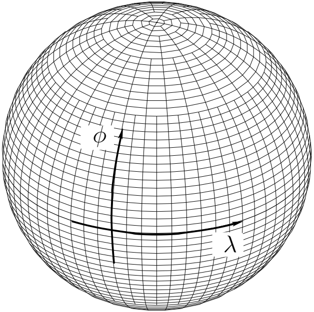
SPHERE
Mon Apr 13 00:03:12 2015

Grid on sphere—the Kurihara Grid

D. J. Williamson–The evolution of dynamical cores for global atmospheric models,
Journal of the meteorological society of Japan 85B (2007)241–269

DISCUSSION: The "POLE PROBLEM"



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DERIVATION OF THE MODEL

INVARIANT FORM

TWO SYSTEMS

Lower-case letters $=$ Inertial system
Capital letters $=$Rotating system .
Time derivatives of vector functions: $\dot{\vec{q}}(t), \dot{\vec{Q}}(t)$.
Connection by **ROTATION MATRIX**

$$\vec{x} = R(t)\vec{X}.$$

$$\dot{\vec{x}} = \frac{d}{dt}R(t)\vec{X} = R(t)(\vec{\Omega} \times \vec{X}).$$

$\vec{\Omega} = \vec{\Omega}(t) =$ angular velocity in the rotating system. It is *constant* (namely, independent of time) in the rotating system.

If $\vec{X}(t)$ represents a moving particle in the rotating system,

$$\dot{\vec{x}} = \frac{d}{dt} R(t)\vec{X} = R(t)(\vec{\Omega} \times \vec{X}) + R(t)(\dot{\vec{X}}).$$

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$$\begin{aligned} \ddot{\vec{x}} &= R(t)\{\vec{\Omega} \times (\vec{\Omega} \times \vec{X}) + \vec{\Omega} \times \dot{\vec{X}} + \vec{\Omega} \times \dot{\vec{X}} + \ddot{\vec{X}}\} \\ &= R(t)\{\vec{\Omega} \times (\vec{\Omega} \times \vec{X}) + 2\vec{\Omega} \times \dot{\vec{X}} + \ddot{\vec{X}}\}. \end{aligned}$$

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Particle of mass m, force $\vec{f}$ (in the inertial system):

$$R(t)(m\ddot{\vec{X}}) = \vec{f} - mR(t)\{\vec{\Omega} \times (\vec{\Omega} \times \vec{X}) + 2\vec{\Omega} \times \dot{\vec{X}}\}.$$

Lagrangian formulation: particle has *unit mass*, and is an element of a fluid continuum moving (approximately) on the spherical surface of the earth S..
$\vec{N}$ = outward unit normal on the sphere S.

TWO "ADDITIONAL FORCES"

CENTRIFUGAL FORCE $\vec{\Omega} \times (\vec{\Omega} \times \vec{X})$

CORIOLIS FORCE $2\vec{\Omega} \times \dot{\vec{X}}$

Velocity $\vec{V} = \dot{\vec{X}}.$

ASSUMPTION I:

There are two body forces acting on the particle:

- $-\vec{G}$ — the gravity force .
- $\vec{H}$ — the hydrostatic force (due to fluid pressure).

Total force (in the inertial system) on the unit mass is

$$\vec{f} = R(t)(-\vec{G} + \vec{H}).$$

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<u>Note</u>: $\vec{X}$ is a three-dimensional vector in the rotational system.
<u>Later:</u> Confine to the sphere $S : r = a$, by assuming that the fluid volume is very "thin"(vertically).

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Some constant $g^* > 0$,

$$-\vec{G} - \vec{\Omega} \times (\vec{\Omega} \times \vec{X}) = -g^* \vec{N}.$$

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MEANING: Earth is not a perfect sphere, the combination of the gravitational and the centrifugal forces can be incorporated into a perfect spherical setting where the "modified" gravitational force is radial.

GEOPHYSICAL LITERATURE: geopotential and the geoid.

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Remain (apart from the modified gravitation):

CORIOLIS FORCE $-2\vec{\Omega} \times \vec{V}$ *HYDROSTATIC FORCE* $\vec{H}$.

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Earth surface $S : r = a$: At every point orthonormal system (fixed in rotational system): unit normal $\vec{N}+$ "tangential plane".

$$\vec{X} = \vec{X}_N + \vec{X}_T$$

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$$(\ddot{\vec{X}})_T = \vec{H}_T - (2\vec{\Omega} \times \vec{V})_T.$$

SHALLOW-WATER MODEL

Incompressible fluid occupies a “thin”, yet varying in depth (and in time) layer above the spherical surface $S : r = a$.

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$Y =$ point on the sphere,
$z =$ vertical distance (along the normal $\vec{N}$) from the surface $S : z = 0$.

$$0 \leq z \leq h(Y, t), \quad Y \in S.$$

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Fluid is incompressible (of unit density).

$h(Y, t) =$ height of column over $Y =$ *(surface) mass density* at Y, t.

ASSUMPTION III:

Tangential velocity $\vec{V}_T = \dot{\vec{X}}_T$ independent of the height z.

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Motion of "surface mass", density h, determined by $\vec{V}_T$.
Conservation of mass:

$$\frac{\partial h}{\partial t}(Y, t) + \nabla_T \cdot (h(Y, t)\vec{V}_T) = 0.$$

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"Surface Lagrangian" derivative:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{V}_T \cdot \nabla_T,$$

$$\frac{dh}{dt}(Y,t) = -h(Y,t)\nabla_T \cdot \vec{V}_T.$$

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$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{V}_T \cdot \nabla_T,$$

$$\frac{dh}{dt}(Y,t) = -h(Y,t)\nabla_T \cdot \vec{V}_T.$$

Total derivative $\frac{d}{dt}$ is a *"surface derivative"*.

ASSUMPTION III:

Tangential velocity $\vec{V}_T = \dot{\vec{X}}_T$ independent of the height z.
Motion of "surface mass", density h, determined by $\vec{V}_T$.
Conservation of mass:

$$\frac{\partial h}{\partial t}(Y, t) + \nabla_T \cdot (h(Y, t)\vec{V}_T) = 0.$$

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Total derivative $\frac{d}{dt}$ is a *"surface derivative"*.
"Convective" part $= \vec{V}_T \cdot \nabla_T = \nabla_{V_T}$, covariant derivative.

ASSUMPTION IV:

Hydrostatic force $\vec{H}$ is the gradient (in the rotating system) of the hydrostatic pressure in the fluid,

$$\vec{H} = -\nabla P.$$

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Normal component of

$$\ddot{\vec{X}} = -\vec{G} + \vec{H} - \vec{\Omega} \times (\vec{\Omega} \times \vec{X}) - 2\vec{\Omega} \times \dot{\vec{X}}.$$

$$(\ddot{\vec{X}})_z = -g^* - \frac{\partial}{\partial z} P - (2\vec{\Omega} \times \vec{V})_z.$$

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This equation is *three-dimensional,* $\{0 \le z \le h(Y, t),\ Y \in S\}$. In particular, the $z-$ component of the Lagrangian derivative

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Traditional treatment:

$$-g^* - \frac{\partial}{\partial z} P = 0, \quad \vec{\Omega}_T = 0.$$

$$V_z(Y, 0, t) \equiv 0 \Rightarrow \left[\dot{\vec{V}}\right]_z (Y, 0, t) \equiv 0.$$

R. Salmon writes: :

“In the traditional approximation, we neglect the horizontal component of the Earth’s rotation vector. This neglect has no convincing general justification; it must be justified in particular cases.”

We only use Assumption IV, in particular P is not assumed to vary linearly with respect to z.

$$V_z(Y, 0, t) \equiv 0 \Rightarrow \left[\dot{\vec{V}}\right]_z(Y, 0, t) \equiv 0.$$

$$V_z(Y, h(Y, t), t) = \frac{dh}{dt}.$$

$$(\dot{\vec{V}})_z = -g^* - \frac{\partial}{\partial z} P - (2\vec{\Omega} \times \vec{V})_z.$$

$$\int_0^{h(Y,t)} \left[\dot{\vec{V}}\right]_z dz = -[g^* + (2\vec{\Omega}_T \times \vec{V}_T)_z] h(Y,t) + P(Y,0,t),$$

$\vec{\Omega}_T$ depends only on Y, $\vec{V}_T$ depends only on (Y, t) (ASSUMPTION III).

$$P(Y,0,t) = [g^* + (2\vec{\Omega}_T \times \vec{V}_T)_z] h(Y,t), \quad Y \in S.$$

EFFECT of ROTATION on HYDROSTATIC PRESSURE!

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BACK TO TANGENTIAL MOTION

$$(\ddot{\vec{X}})_T = (\dot{\vec{V}})_T = \vec{H}_T - (2\vec{\Omega} \times \vec{V})_T.$$

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INVARIANT SHALLOW-WATER EQUATIONS ON THE SPHERE

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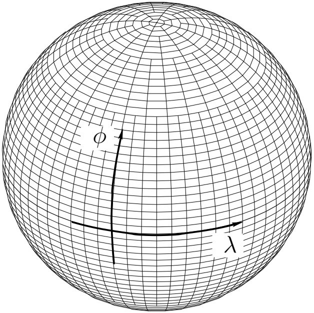
Compare Equator and Poles !

THE SW EQUATIONS –SPHERICAL COORDINATES

$$-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}, \quad 0 \leq \lambda \leq 2\pi.$$

$$\frac{\partial h}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial h}{\partial \lambda} + \frac{v}{a}\frac{\partial h}{\partial \phi} + \frac{h}{a\cos\phi}\left(\frac{\partial u}{\partial \lambda} + \cos\phi\frac{\partial v}{\partial \phi}\right) = \frac{hv\sin\phi}{a\cos\phi}.$$



$$\delta = 0 \Rightarrow \text{set } \vec{\Omega}_T = 0, \text{ otherwise } \delta = 1.$$

$$\frac{\partial u}{\partial t} + \frac{u - 2\delta\Omega h \cos\phi}{a\cos\phi}\frac{\partial u}{\partial \lambda} + \frac{v}{a}\frac{\partial u}{\partial \phi} + \frac{g^* - 2\delta\Omega u \cos\phi}{a\cos\phi}\frac{\partial h}{\partial \lambda}$$

$$= v\sin\phi\left\{\frac{u}{a\cos\phi} + 2\Omega\right\} \quad ,$$

$$\frac{\partial v}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial v}{\partial \lambda} + \frac{v}{a}\frac{\partial v}{\partial \phi} + \frac{g^* - 2\delta\Omega u\cos\phi}{a}\frac{\partial h}{\partial \phi} - \frac{2\delta\Omega h\cos\phi}{a}\frac{\partial u}{\partial \phi}$$

$$+2\Omega\sin\phi\frac{\delta h u}{a} = -\frac{u^2}{a\cos\phi}\sin\phi - 2\Omega u\sin\phi \quad .$$

THE SPLIT SCHEME WITH SOURCE TERMS

$$\psi_t = A[\psi] + B[\psi] + f(\cdot, \psi),$$

Consider first the homogeneous evolution

$$\psi_t = A[\psi] + B[\psi],$$

$$\psi(t) = \mathfrak{L}_{AB}(t)\psi_0.$$

Nonhomogeneous system: A, B are **linear,** but not necessarily commuting , the solution is expressed by the Duhamel principle

$$\psi(t) = \mathfrak{L}_{AB}(t)\psi_0 + \int_0^t \mathfrak{L}_{AB}(t-s)[f(\cdot, \psi(s))]ds.$$

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Assuming existence of a discrete operator ("scheme") $\mathfrak{L}_{AB}^{disc}(k)$, time step $k > 0$, that approximates $\mathfrak{L}_{AB}(k)$:
$\psi(t) = \mathfrak{L}_{AB}\psi_0$ solution to the homogeneous equation. Fix $T > 0$.
Then there exist a constant $C > 0$ and an integer $j \geq 1$, such that

$$\|\mathfrak{L}_{AB}^{disc}(k)[\psi(t)] - \psi(t+k)\| \leq Ck^{j+1}, \quad 0 \leq t \leq T.$$

DISCRETIZATION OF NONHOMOGENEOUS EQUATION

Splitting with two "generators", $A+B$ and f.

$$(i) \qquad \psi_t = A[\psi] + B[\psi],$$
$$(ii) \qquad \psi_t = f(\cdot, \psi).$$

$$\psi_{tt} = f'_\psi(\cdot, \psi(t)) \cdot \psi_t = f'_\psi(\cdot, \psi(t)) \cdot f(\cdot, \psi(t)).$$

Discretization of (ii):

$$\mathfrak{M}^{disc}(k)[\psi(t)] = \psi(t) + kf(\cdot, \psi(t)) + \frac{k^2}{2} f'_\psi(\cdot, \psi(t)) \cdot f(\cdot, \psi(t)).$$

SUMMARY: A discrete operator $\Gamma^{disc}(k)$ for the approximation of the full system over the time interval $[t, t+k]$ is given by

$$\Gamma^{disc}(k) = \mathfrak{M}^{disc}(k)\, \mathfrak{L}^{disc}_{AB}(k).$$

A SCALAR MODEL ON MANIFOLDS

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- No SELF-SIMILAR SOLUTIONS—Riemann Problems are not defined.
- Waves produce multiple "recurring" interactions.

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A **flux** on a manifold (M^n, g) is a vector field $f = f_x(\overline{u})$ depending upon the parameter $\overline{u}$ (the dependence in both variables being smooth).

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$$\partial_t u + \nabla_g \cdot (f_x(u)) = 0,$$

Unknown: scalar-valued function $u = u(t, x)$.
$\nabla_g \cdot (f_x(u))$ for fixed t, on vector field $x \hookrightarrow f_x(u(t, x)) \in T_x M$.

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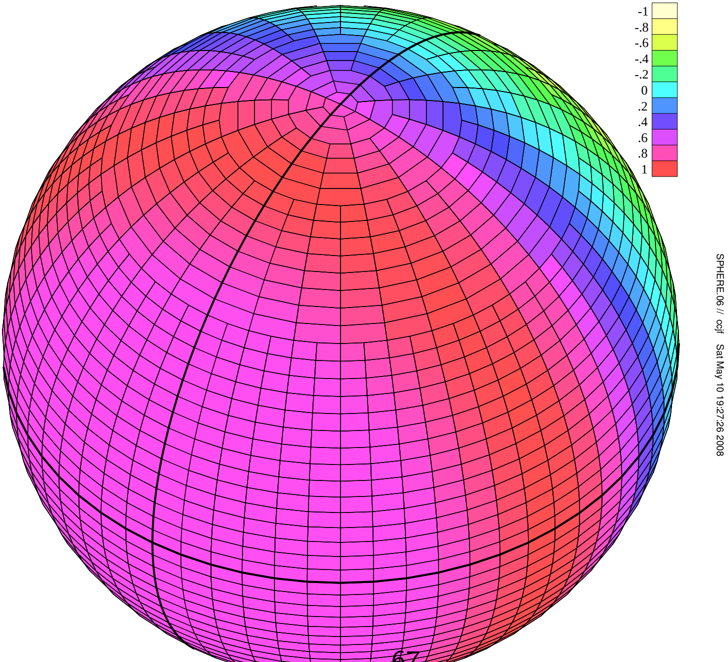
A flux is called **geometry-compatible** if it satisfies the divergence-free condition

$$\nabla \cdot f_x(\overline{u}) = 0, \qquad \overline{u} \in \mathbf{R},\ x \in M.$$

CONFINED SOLUTION



The regularized initial-value problem

An initial data $u_0 \in BV(M; dV_g)$, find a solution $u^\varepsilon = u^\varepsilon(t,x)$ to:

$$\partial_t u^\varepsilon + \mathrm{div}_g \left(f_x(u^\varepsilon) \right) = \varepsilon \, \Delta_g u^\varepsilon, \qquad x \in M, \; t \geq 0,$$

$$u^\varepsilon(0,x) = u_0^\varepsilon(x), \quad x \in M,$$

where Δ_g denotes the Laplace operator on the manifold M,

$$\begin{aligned} \Delta_g v &:= \nabla_g \cdot \nabla_g v \\ &= g^{ij} \Big(\frac{\partial^2 v}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial v}{\partial x^k} \Big). \end{aligned}$$

$u_0^\varepsilon : M \to \mathbf{R}$ is a sequence of smooth functions satisfying

$$\begin{aligned} & \|u_0^\varepsilon\|_{L^p(M)} \leq \|u_0\|_{L^p(M)}, \qquad p \in [1, \infty], \\ & TV(u_0^\varepsilon) \leq TV(u_0), \\ & \sup_{0<\varepsilon<1} \varepsilon \, \|u_0^\varepsilon\|_{H^2(M; dV_g)} < \infty, \\ & u_0^\varepsilon \to u_0 \quad \text{a.e. on } M. \end{aligned}$$

REGULARIZED PROBLEM

(Ben-Artzi and LeFloch, 2007)

THEOREM: Let $f = f_x(\overline{u})$ be a geometry-compatible flux on (M, g). Given any initial data $u_0^\varepsilon \in C^\infty(M)$ satisfying the above conditions there exists a unique solution $u^\varepsilon \in C^\infty(\mathbf{R}_+ \times M)$ to the initial value problem . Moreover, for each $1 \leq p \leq \infty$ the solution satisfies

$$\|u^\varepsilon(t)\|_{L^p(M;dV_g)} \leq \|u^\varepsilon(t')\|_{L^p(M;dV_g)}, \qquad 0 \leq t' \leq t$$

and, for any two solutions u^ε and v^ε,

$$\|v^\varepsilon(t) - u^\varepsilon(t)\|_{L^1(M;dV_g)} \leq \|v^\varepsilon(t') - u^\varepsilon(t')\|_{L^1(M;dV_g)}, \quad 0 \leq t' \leq t.$$

In addition, for every convex entropy/entropy flux pair (U, F_x) the solution u^ε satisfies the entropy inequality

$$\partial_t U(u^\varepsilon) + \mathrm{div}_g \left(F_x(u^\varepsilon)\right) \leq \varepsilon \, \Delta_g U(u^\varepsilon).$$

ENTROPY SOLUTION

(Ben-Artzi and LeFloch, 2007)
CORRECTION: Lengeler and Müller 2013
THEOREM: Let $f = f_x(\overline{u})$ be a geometry-compatible flux on (M, g). Given any bounded initial function $u_0 \in BV(\mathbf{M}^n; dV_g)$ there exists an **entropy solution** $u \in L^\infty(\mathbf{R}_+ \times \mathbf{M}^n)$ to the initial value problem , so that

$$\|u(t)\|_{L^p(\mathbf{M}^n; dV_g)} \leq \|u_0\|_{L^p(\mathbf{M}^n; dV_g)}, \quad t \geq 0, \; p \in [1, \infty].$$

For some constant $C_1 > 0$ depending on $\|u_0\|_{L^\infty(M)}$ and the Ricci tensor

$$\begin{aligned} & TV(u(t)) \leq e^{C_1 t} (1 + TV(u_0)), \qquad t \in \mathbf{R}_+, \\ & \|u(t) - u(t')\|_{L^1(M; dV_g)} \leq C_1 \, TV(u_0) \, |t - t'|, \quad 0 \leq t' \leq t. \end{aligned} \tag{1}$$

Definition

Let $f = f_x(\overline{u})$ be a geometry-compatible flux on (M, g). Given any initial condition $u_0 \in L^\infty(M)$, a measure-valued map $(t, x) \in \mathbf{R}_+ \times M \mapsto \nu_{t,x}$ is called an **entropy measure-valued solution** *to the initial value problem if, for every convex entropy/entropy flux pair (U, F_x) ,*

$$\iint_{\mathbf{R}_+ \times M} \Big(\langle \nu_{t,x}, U \rangle \, \partial_t \theta(t,x) + g_x\big(\langle \nu_{t,x}, F_x \rangle, grad_g \, \theta(t,x)\big) \Big) \, dV_g(x) dt + \int_M U(u_0(x)) \, \theta(0,x) \, dV_g(x) \geq 0, \tag{2}$$

for every smooth function $\theta = \theta(t,x) \geq 0$ compactly supported in $[0, +\infty) \times M$.

THEOREM

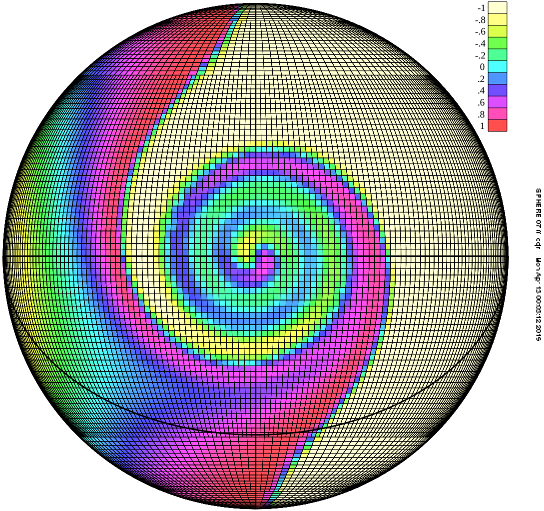
(Well-posedness theory in the measure-valued class for geometry-compatible conservation laws.)

(Ben-Artzi and LeFloch 2006)
Let $f = f_x(\overline{u})$ be a geometry-compatible flux on (M, g), and let $u_0 \in L^\infty(M)$. Then there exists a unique entropy measure-valued solution ν to the initial value problem . For almost every (t, x), the measure $\nu_{t,x}$ is a Dirac mass, i.e. of the form

$$\nu_{t,x} = \delta_{u(t,x)},$$

where the function $u \in L^\infty(\mathbf{R}_+ \times M)$. Moreover, the initial data is attained in the strong sense

$$\limsup_{t \to 0+} \int_M |u(t,x) - u_0(x)| \, dV_g(x) = 0. \tag{3}$$



-1
-.8
-.6
-.4
-.2
0
.2
.4
.6
.8
1
SPHERE

THANK YOU!