Utilizing Geometry of Smoothness-Increasing-Accuracy-Conserving (SIAC) filters for reduced errors

Joint work with

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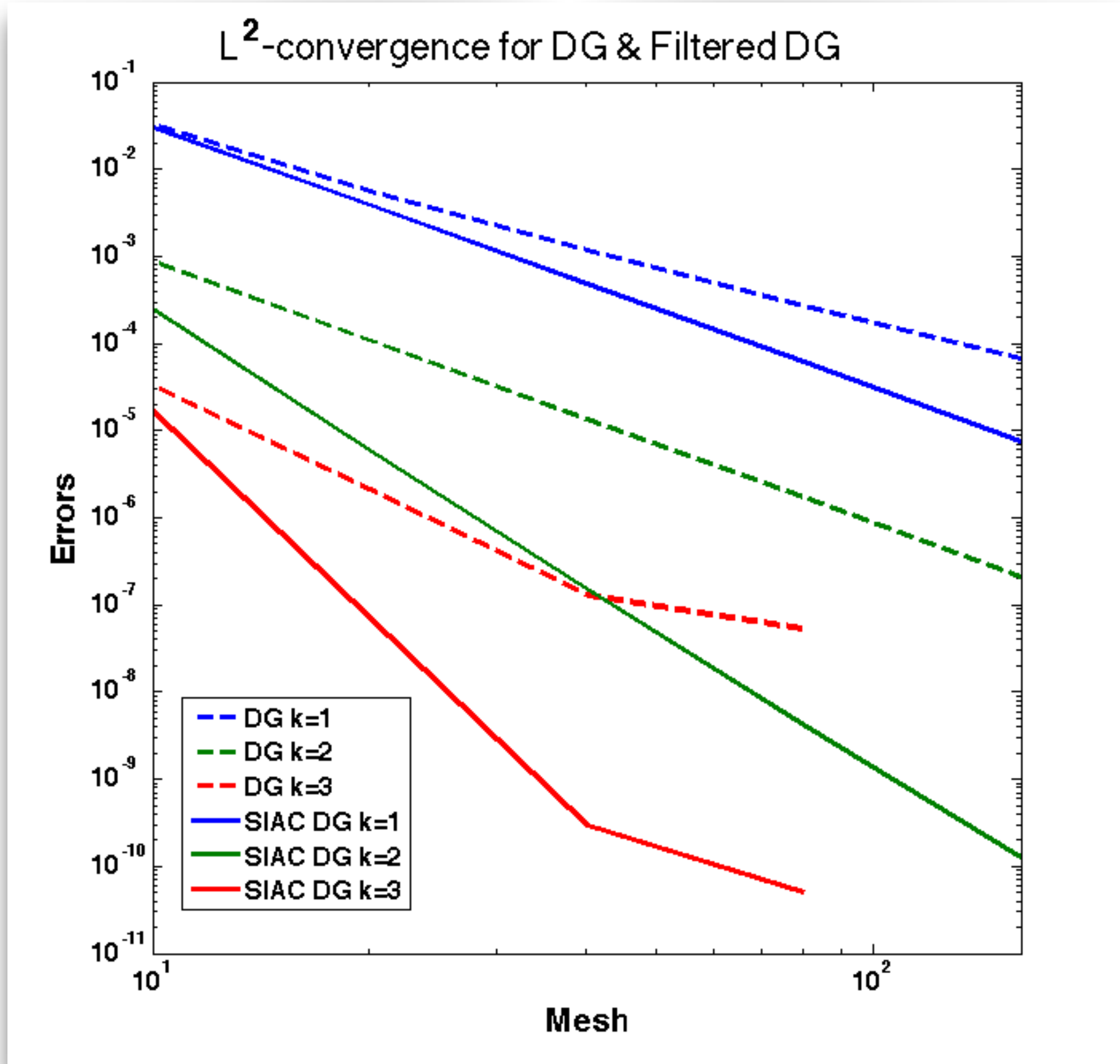
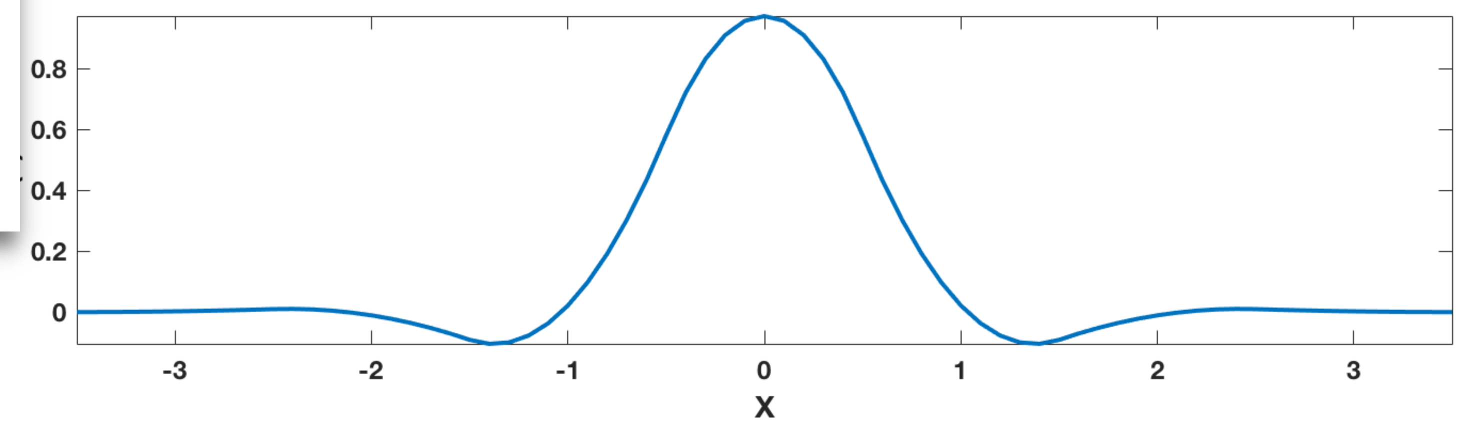
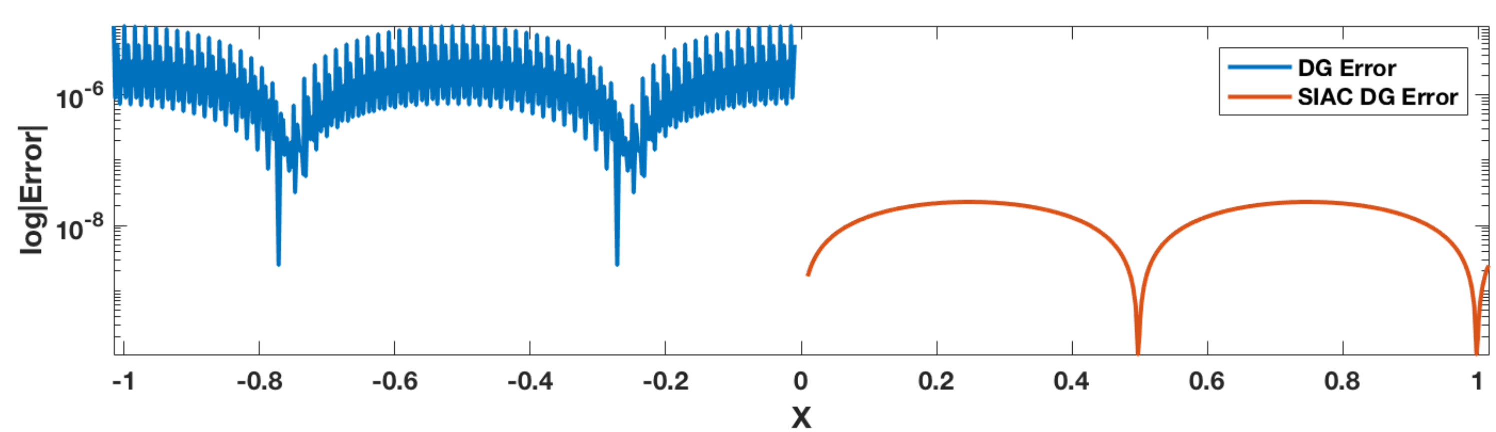
SMOOTHNESS-INCREASING ACCURACY-CONSERVING (SIAC) FILTER



SIAC filtering allows:

- Extract global superconvergence
- Create globally smooth approximations



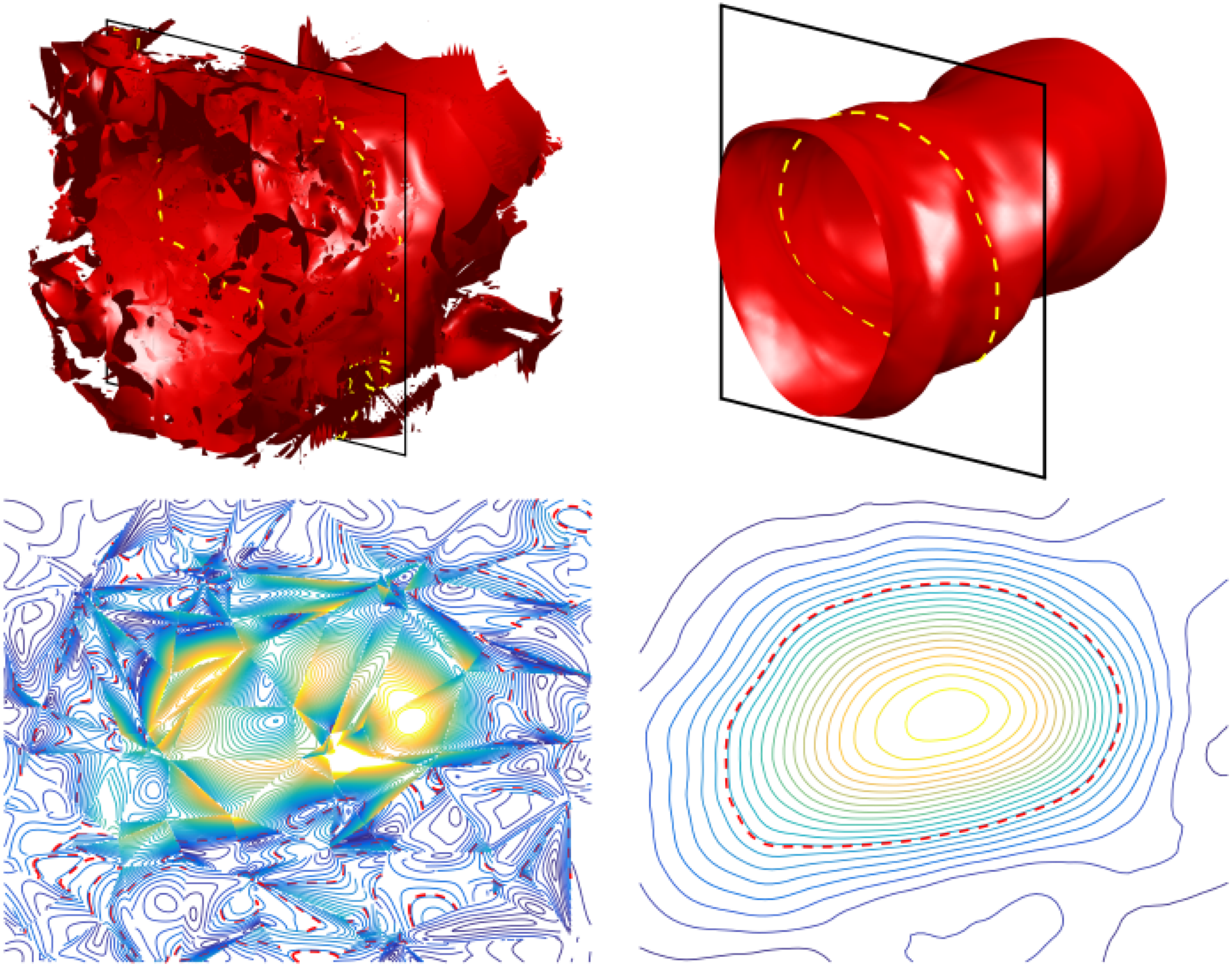


Approximation order: *p+1*

SIAC DG order: *2p+1*

APPLICATIONS: FLOW VISUALIZATION

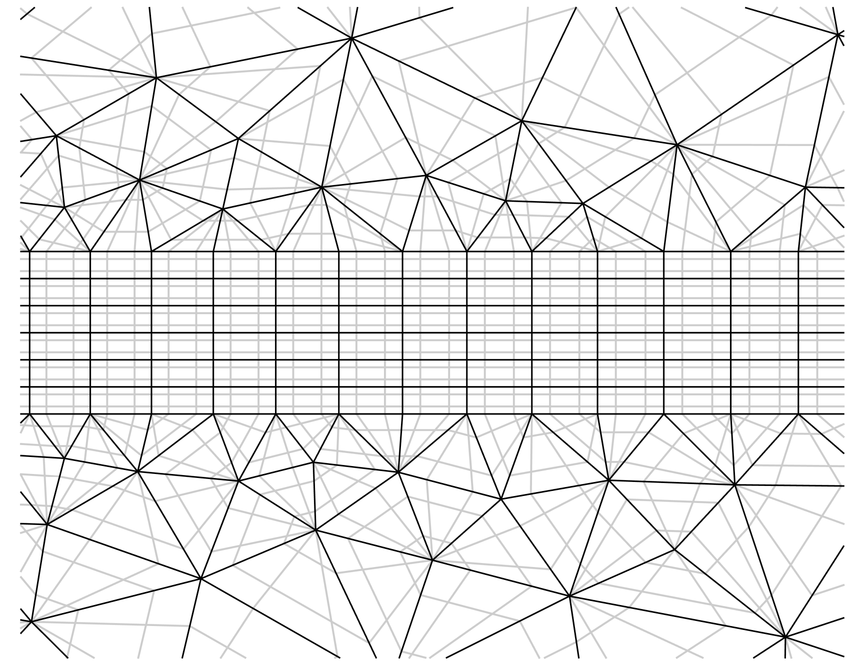
Visualizing Vorticity (from NACA wing simulation)

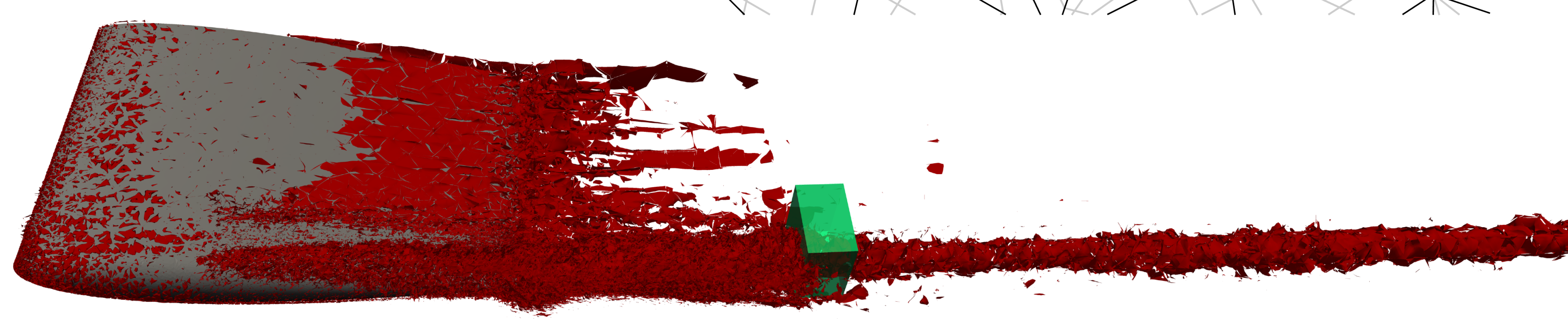


Jallepelli, Docampo, Ryan, Haimes, and Kirby, IEEE TVCG, 2018

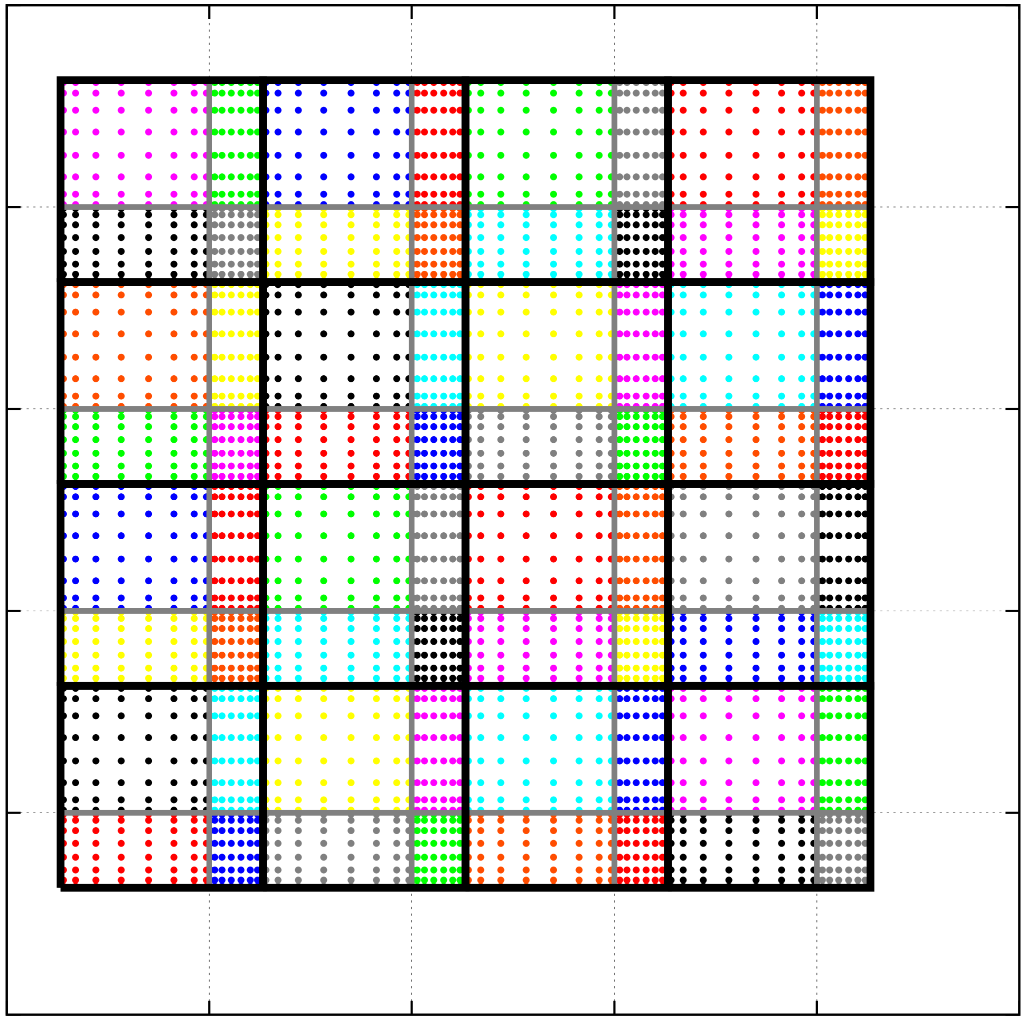
DATA INFORMATION

- Re = 1.2×10^6, 12 degree angle of attack
- Results from Nektar++:
  - continuous Galerkin (cG) discretization
  - *p=5* polynomials
  - *211180* tetrahedra
  - *38680* prisms
- Box is sampled and vorticity is computed at a resolution of *90 x 90 x 90.*

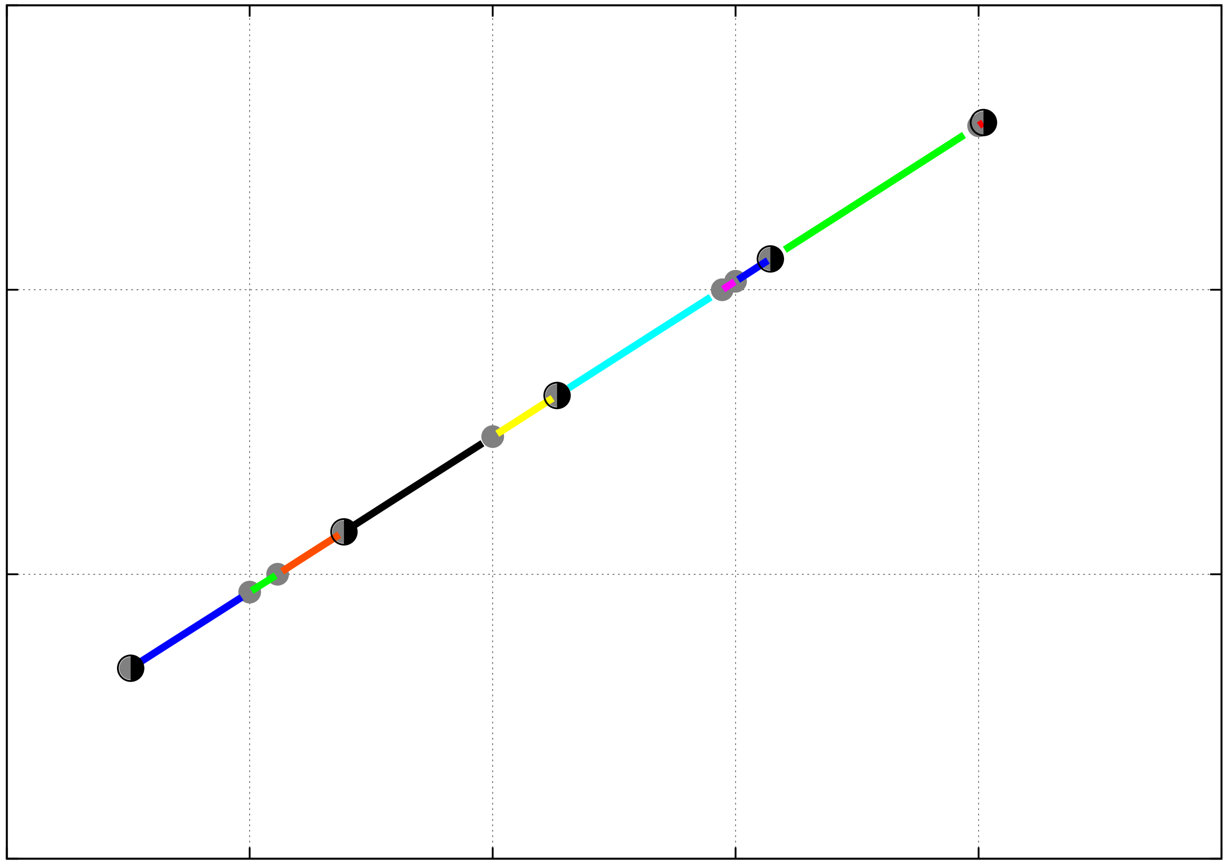




2D SIAC FILTER: COMPUTATIONAL FOOTPRINT



Tensor Product Filter



1D Line SIAC Filter

OUTLINE

- Background
  - Convergence properties of Discontinuous Galerkin method
  - Smoothness-Increasing Accuracy-Conserving (SIAC) filter
- Divided Difference Estimates
  - Line SIAC filter
- Numerical results
- Conclusion & Future Work

DISCONTINUOUS GALERKIN: CONVERGENCE PROPERTIES

For linear hyperbolic equations over a regular grid:

- In L^2:

$$\|u - u_h\|_0 \leq Ch^{p+1}$$

- Outflow edge:

$$|(u - u_h)(x_{j+1/2})| \leq Ch^{2p+1}$$

- Negative-Order norm:

$$\|\partial_h^\alpha (u - u_h)\|_{-(p+1)} = \sup_{\Phi \in \mathcal{C}_0^\infty} \frac{(\partial_h^\alpha (u - u_h), \Phi)}{\|\Phi\|_{p+1}} \leq Ch^{2p+1}$$

Extracting Superconvergence

Smoothness-Increasing Accuracy-Conserving (SIAC) Filters

SIAC FILTERED DG

SIAC filtered solution:

$$u_h^*(x,t) = \frac{1}{H}\int_{\mathbb{R}} K\left(\frac{x-y}{H}\right) u_h(y,t)\,dy$$

SIAC filtered error:

$$\text{(Uniform)} \qquad \|(u - K_h^{(2p+1,p+1)} * u_h)(T)\|_0 \leq C h^{2p+1}$$

$$\text{(Non-uniform)} \qquad \|(u - K_H^{(2p+1,p+1)} * u_h)(T)\|_0 \leq C h^{\frac{2}{3}(2p+1)}$$

Mock & Lax (1978)
Bramble & Schatz, *Math. Comp* (1977)
Cockburn, Luskin, Shu, & Süli, *Math. Comp* (2003)
(L-inf estimates) Ji, Xu, Ryan, Math. Comp. (2012)

SIAC KERNEL

- SIAC kernel:
  - Linear combination of B-splines of order $m+1$.

$$K(x) = \sum_{\gamma=-r}^{r} c_\gamma \psi^{(m+1)}(x-\gamma)$$

Chosen to maintain $2r$ moments

B-spline chosen for desired smoothness

- Filter width: $(2r+m+1)H$, where H is the scaling (generally the mesh size).
- Alternatively: Can choose coefficients to satisfy data requirements.

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SIAC KERNEL: FOURIER SPACE

- In physical space, the filter is

$$K(x) = \sum_{\gamma=-r}^{r} c_\gamma \psi^{(m+1)}(x-\gamma)$$

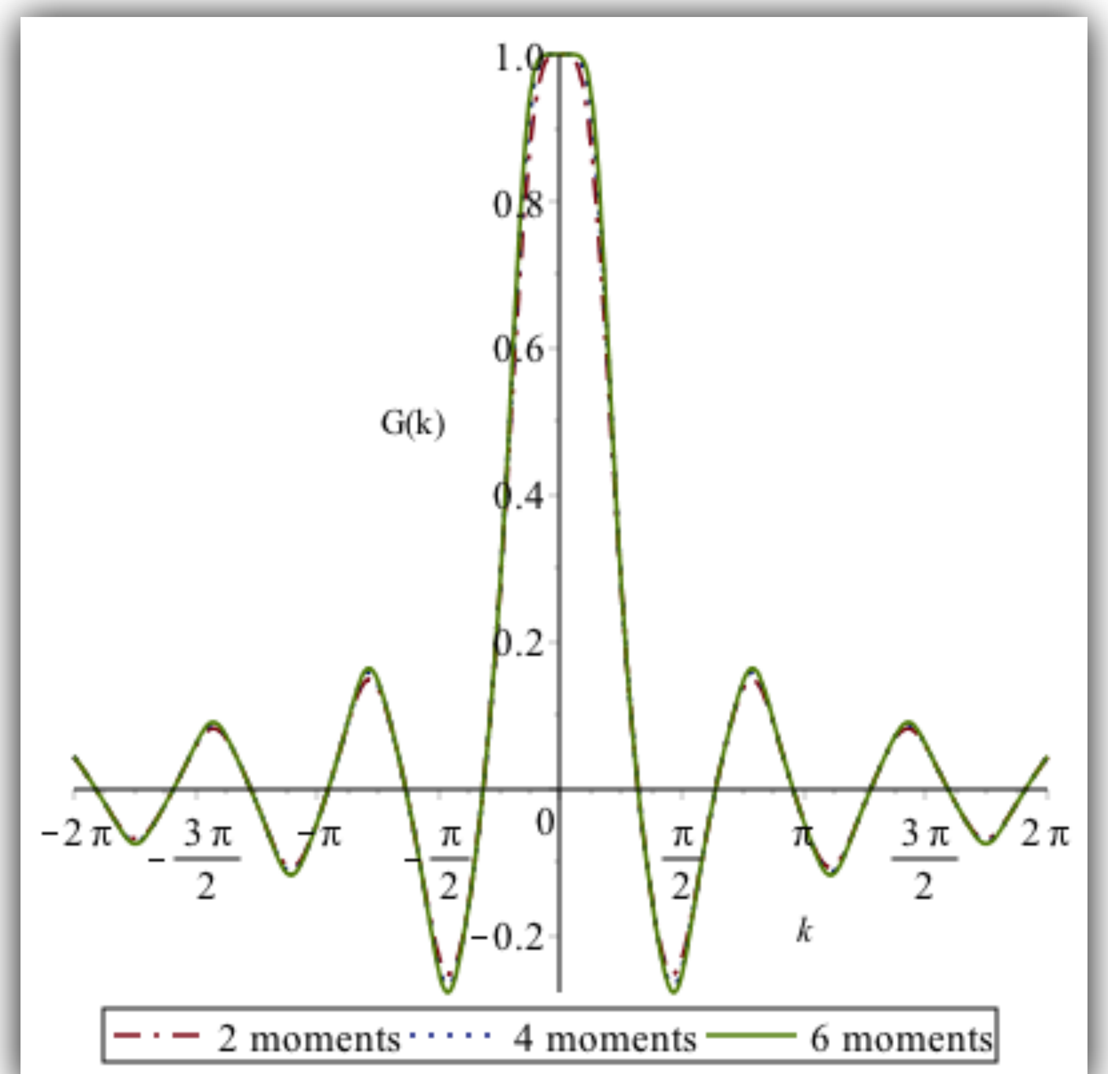
- In Fourier space this is:

$$\hat{K}(k) = \left(\frac{\sin(k\pi)}{k\pi}\right)^{m+1} \left(c_0 + 2\sum_{\gamma=0}^{r} c_\gamma \cos(\gamma k \pi)\right)$$

Thomee, *Math. Comp.* (1977)
Ji & Ryan, *ICOSAHOM 2014 Proceedings*

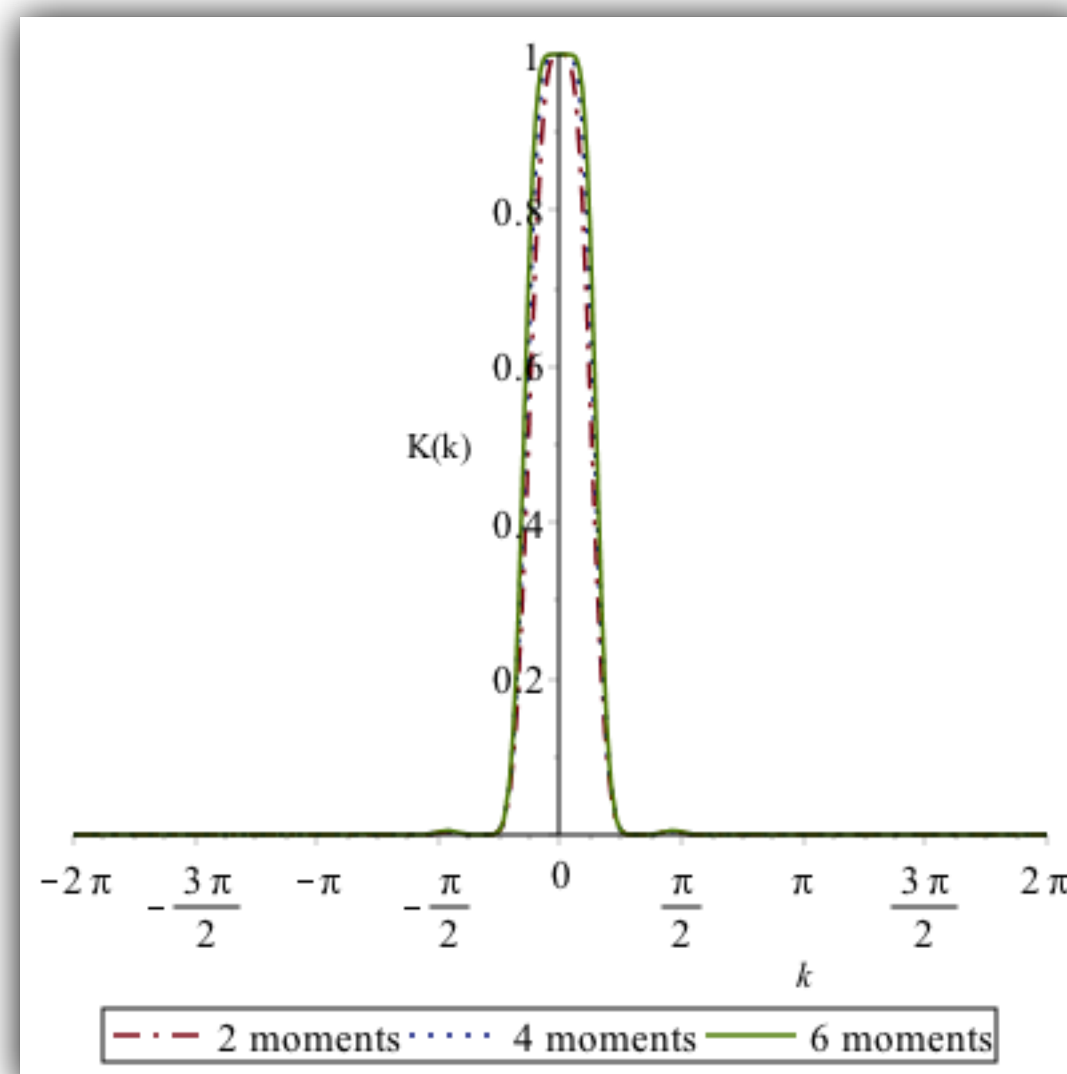
SIAC KERNEL: FOURIER SPACE

SIAC filter: first-order B-spline (top hat function).



Plot of full kernel in Fourier space for preserving 2, 4 and 6 moments.

SIAC filter: fourth-order B-spline.



Plot of full kernel in Fourier space for preserving 2, 4 and 6 moments.

TYPICAL PARAMETER CHOICE

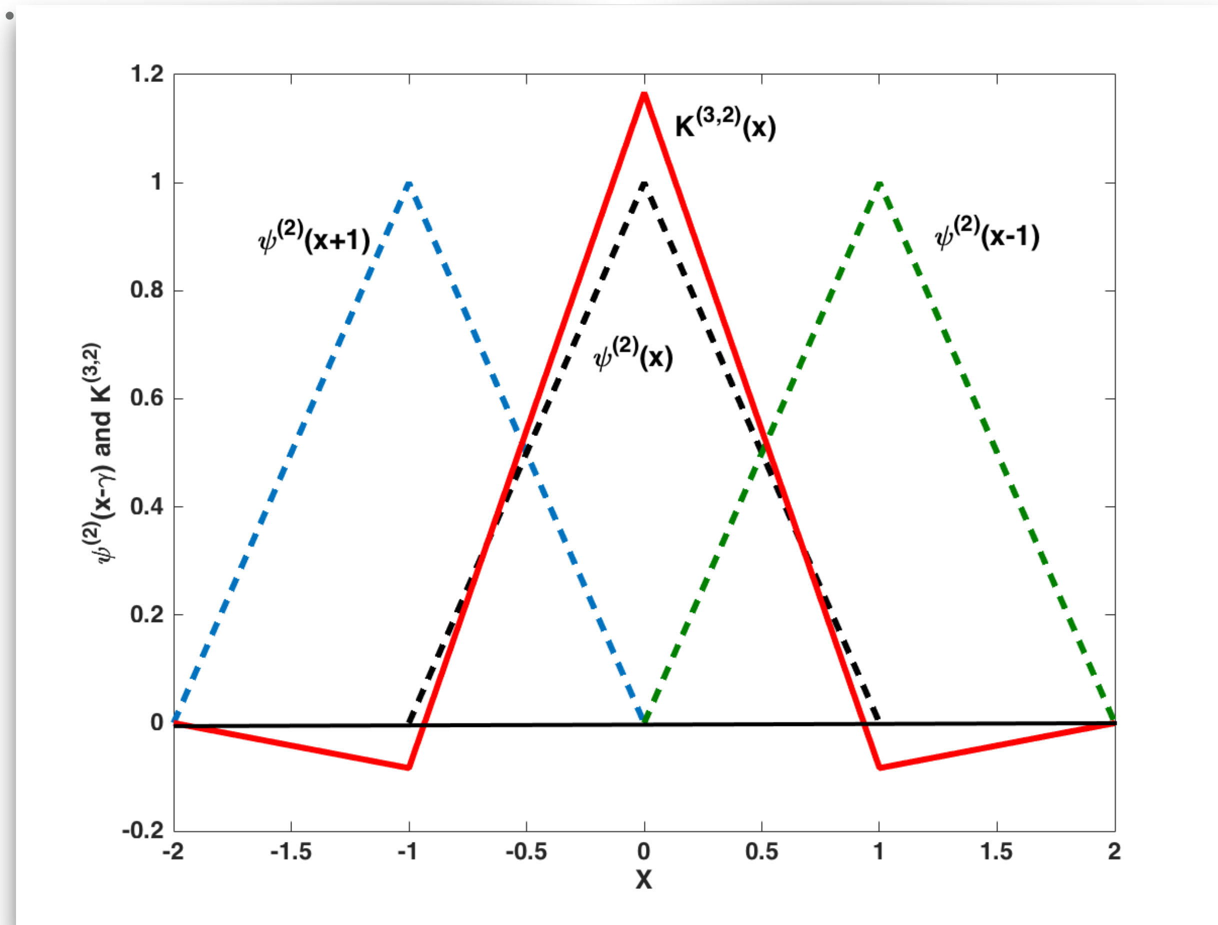
$$u_h^*(x,t) = \frac{1}{H}\int_{\mathbb{R}} K\left(\frac{x-y}{H}\right) u_h(y,t)\, dy$$

$H < \Delta x$
Order ***p+1***

$H = \Delta x$
up to
Order ***2p+1***

p	Number B-Splines	B-Splne Order	Number elements	Possible accuracy
1	3	2	5	3
2	3	1	3	3
	5	3	7 —> 9	5
	5	1	7	5
3	3	1	4 —> 5	3
	7	4	11	7
	7	1	8 —> 9	7
	5	1	6 —> 7	5

SIAC KERNEL

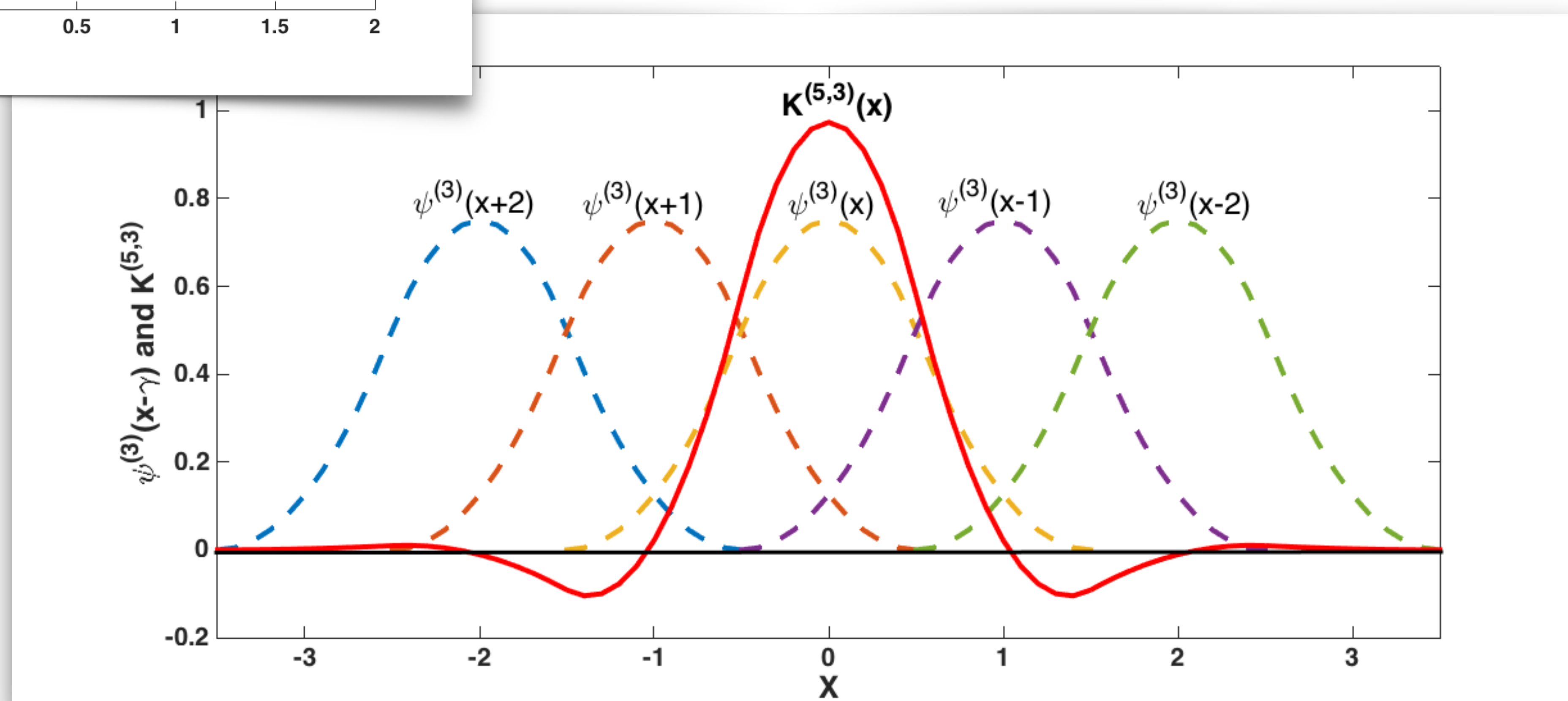


p=1 Kernel:

- Preserves 2 moments
- Continuous
- Support of 5 elements

p=2 Kernel:

- Preserves 4 moments
- C^1 continuity
- Support of 7 elements



SIAC KERNEL: MULTIPLE DIMENSIONS

- The post-processed solution:

$$u_h^*(\bar{\mathbf{x}}, t) = \frac{1}{H_d} \int_{\mathbb{R}^d} K\left(\frac{\bar{x_1} - x_1}{\Delta x_1}\right) K\left(\frac{\bar{x_2} - x_2}{\Delta x_2}\right) \cdots K\left(\frac{\bar{x_d} - x_d}{\Delta x_d}\right) u_h(\mathbf{x}, t)\, d\mathbf{x}$$

where $H_d = \Delta x_1 \Delta x_2 \cdots \Delta x_d$

- The post-processing kernel is the same in each dimension:

$$K(x) = \sum_{\gamma=-r}^{r} c_\gamma \psi^{(m+1)}(x - \gamma)$$

SIAC FILTER: ERROR ESTIMATES

Through convolution, we can obtain a higher order accurate solution:

$$\|u - u_h^*\|_0 \leq \underbrace{\|u - u^*\|_0}_{\text{Filter Error}} + \underbrace{\|K_H * (u - u_h)\|_0}_{\text{Discretization Error}}$$

Error in filtered solution

Determined by
number of moments ($2r$) the filter preserves
Ch^{2r+1}

Determined by
- Numerical scheme
- Choice of kernel function
- (Dual problem of PDE)
- Ch^s

SIAC FILTER: ERROR ESTIMATES

Second term

Using properties of B-splines and convolution

$$\|K_H * (u - u_h)\|_0 \leq \sum_{|\alpha| \leq m+1} \|D^\alpha (K_H * (u - u_h))\|_{-(m+1)}$$

$$\leq \sum_{|\alpha| \leq m+1} \|K\|_1 \|\partial_H^\alpha (u - u_h)\|_{-(m+1)}$$

Properties of the scheme/Equation

where

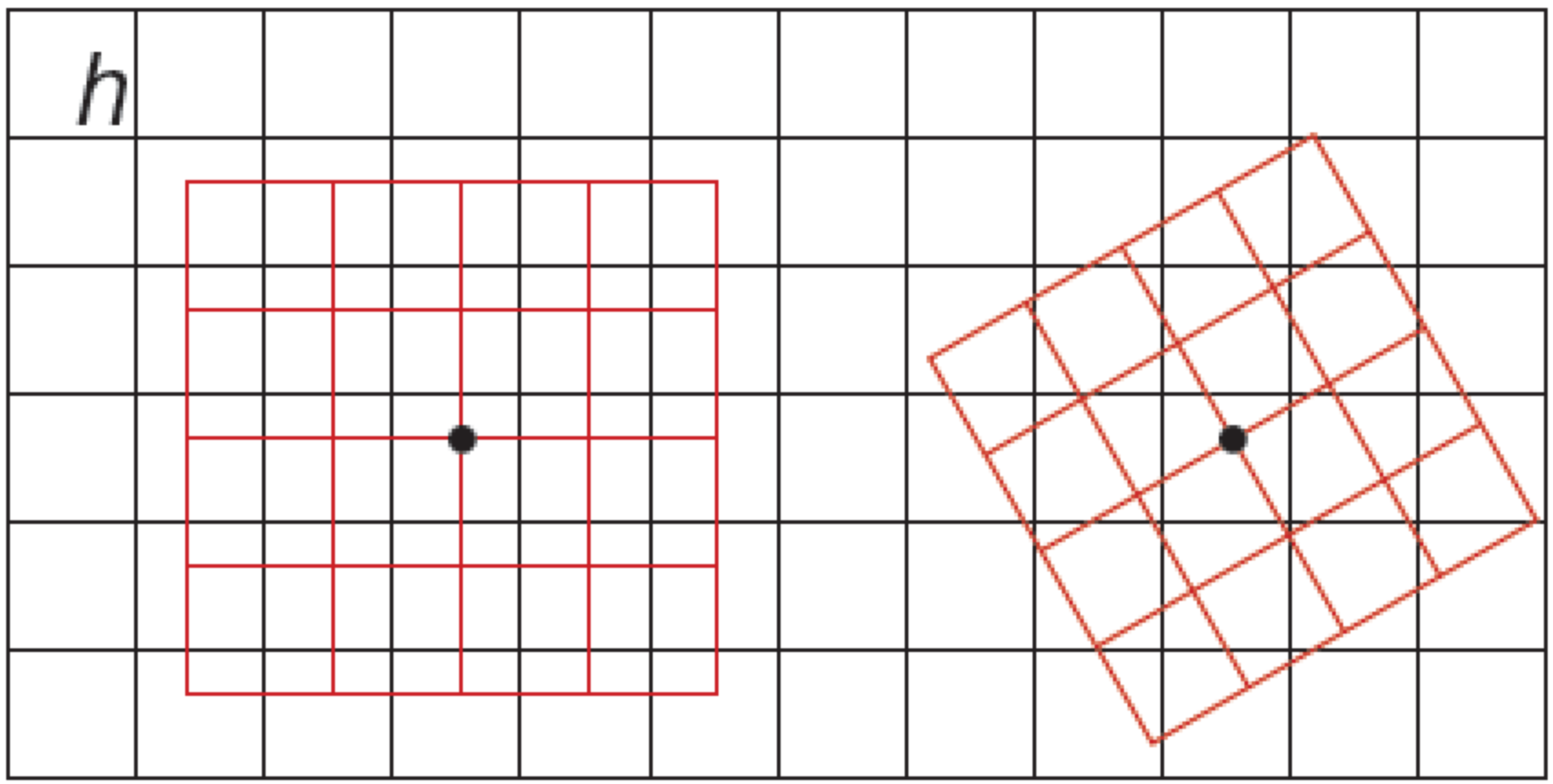
$$\|\partial_h^\alpha (u - u_h)\|_{-(p+1)} \leq C h^{2p+1}$$

Question:

How can we use a 1D filter for 2D data?

LINE SIAC FILTER FOR MULTI-DIMENSIONAL FILTERING

- Cartesian-aligned filter vs. Rotated filter



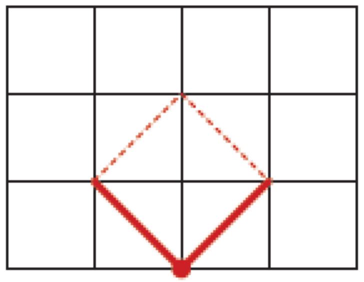
$H = h$ $\qquad$ $H = h\cos\theta$

h is the uniform DG element size, *H* is the kernel scaling

SIAC FILTER: DIVIDED DIFFERENCE ESTIMATES

- We need to worry about:

$$\|\partial_H^{\alpha}(u-u_h)\|_{-(m+1)}$$



- Requires:
  - Relating coordinate-aligned derivatives with arc-length derivatives

$$D^{\alpha}\widetilde{\psi}^{(m+1)}(x,y) = \frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}}\frac{\partial^{\alpha_2}}{\partial y^{\alpha_2}}\widetilde{\psi}^{(m+1)}(x,y)$$

$$= \cos^{\alpha_1}\theta\,\sin^{\alpha_2}\theta\,\frac{d^{|\alpha|}}{dt^{|\alpha|}}\psi^{(m+1)}(t)$$

$$= \cos^{\alpha_1}\theta\,\sin^{\alpha_2}\theta\,\partial_H^{|\alpha|}\psi^{(m+1-\alpha)}$$

SIAC FILTER: DIRECTIONAL DIVIDED DIFFERENCE ESTIMATES

- Need to relate directional divided differences to coordinate-aligned divided differences.
- Direction vector: $\boldsymbol{u}=(u_x,u_y)$.
- Scaled directional divided difference with respect to $\mathbf{u}$:

$$\partial_{\mathbf{u},H} f(t) = \frac{1}{H}\left(f\left(x+\frac{H}{2}u_x, y+\frac{H}{2}u_y\right) - f\left(x-\frac{H}{2}u_x, y-\frac{H}{2}u_y\right)\right)$$

$$= \partial_{u_x,H} f\left(x, y+\frac{H}{2}u_y\right) + \partial_{u_y,H} f\left(x-\frac{H}{2}u_x, y\right).$$

- α–th directional divided difference:

$$\partial^{\alpha}_{\mathbf{u},H} f(x,y) = \partial_{\mathbf{u},H}\left(\partial^{\alpha-1}_{\mathbf{u},H} f(x,y)\right), \quad \alpha > 1.$$

SIAC FILTER: DIVIDED DIFFERENCE ESTIMATES

- We can relate the directional divided difference to the coordinate aligned divided difference

$$D^{\alpha}\psi^{(\ell)}(t) = (\cos\theta)^{\alpha_x}(\sin\theta)^{\alpha_y}\,\partial_h^{\alpha}\,\psi^{(\ell-\alpha)}(\mathbf{x})$$

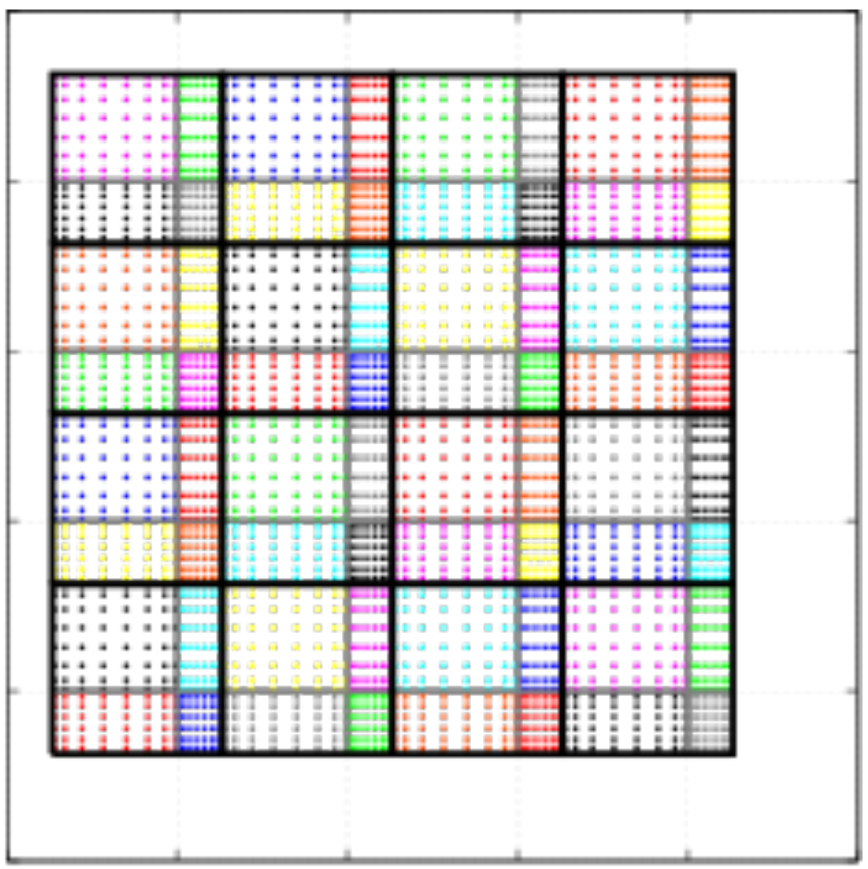
coordinate aligned

- As long as θ ≠ 0, π/2, then we have **superconvergence!**
  - Leads to a *reduced error constant.*

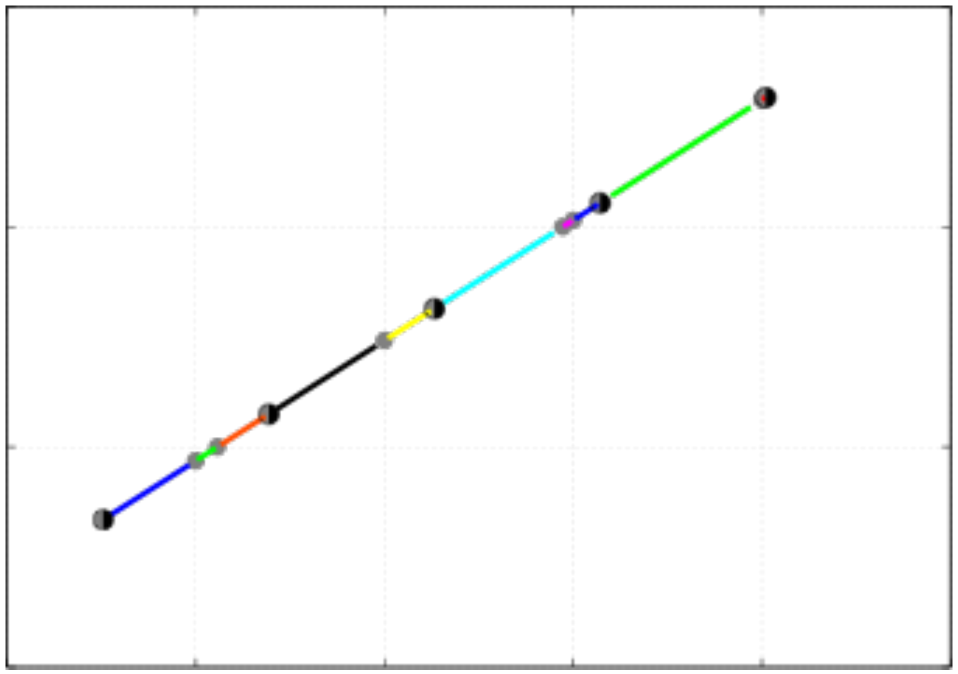
$$\|K_H * (u-u_h)\|_0 \le \cos^{\alpha_1}\theta\,\sin^{\alpha_2}\theta\, C \sum_{|\alpha|\le m+1} \|K_H\|_1\,\|\partial_H^{\alpha}(u-u_h)\|_{-(m+1)}$$

LINE SIAC FILTER FOR MULTI-DIMENSIONAL FILTERING

- Reducing the support to a line: axis-aligned vs. rotated



Tensor Product SIAC filter

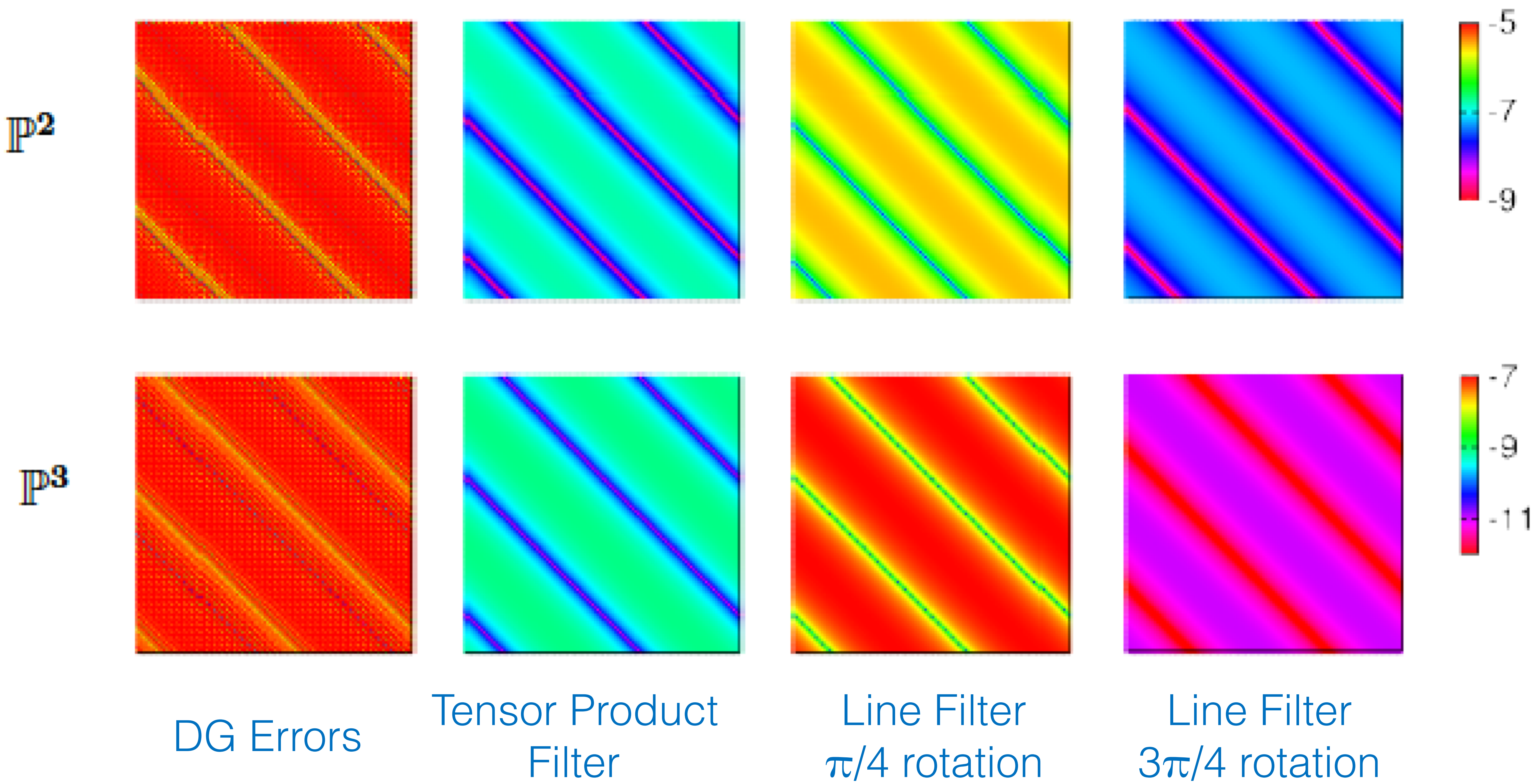


Line SIAC filter

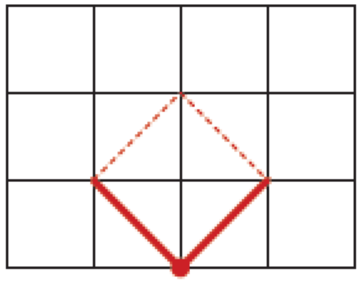
LINE SIAC FILTER FOR MULTI-DIMENSIONAL FILTERING

$$u_t + u_x + u_y = 0, \quad u(x, y, 0) = \sin(x + y)$$

SISC (2017)



LINE SIAC FILTER FOR MULTI-DIMENSIONAL FILTERING

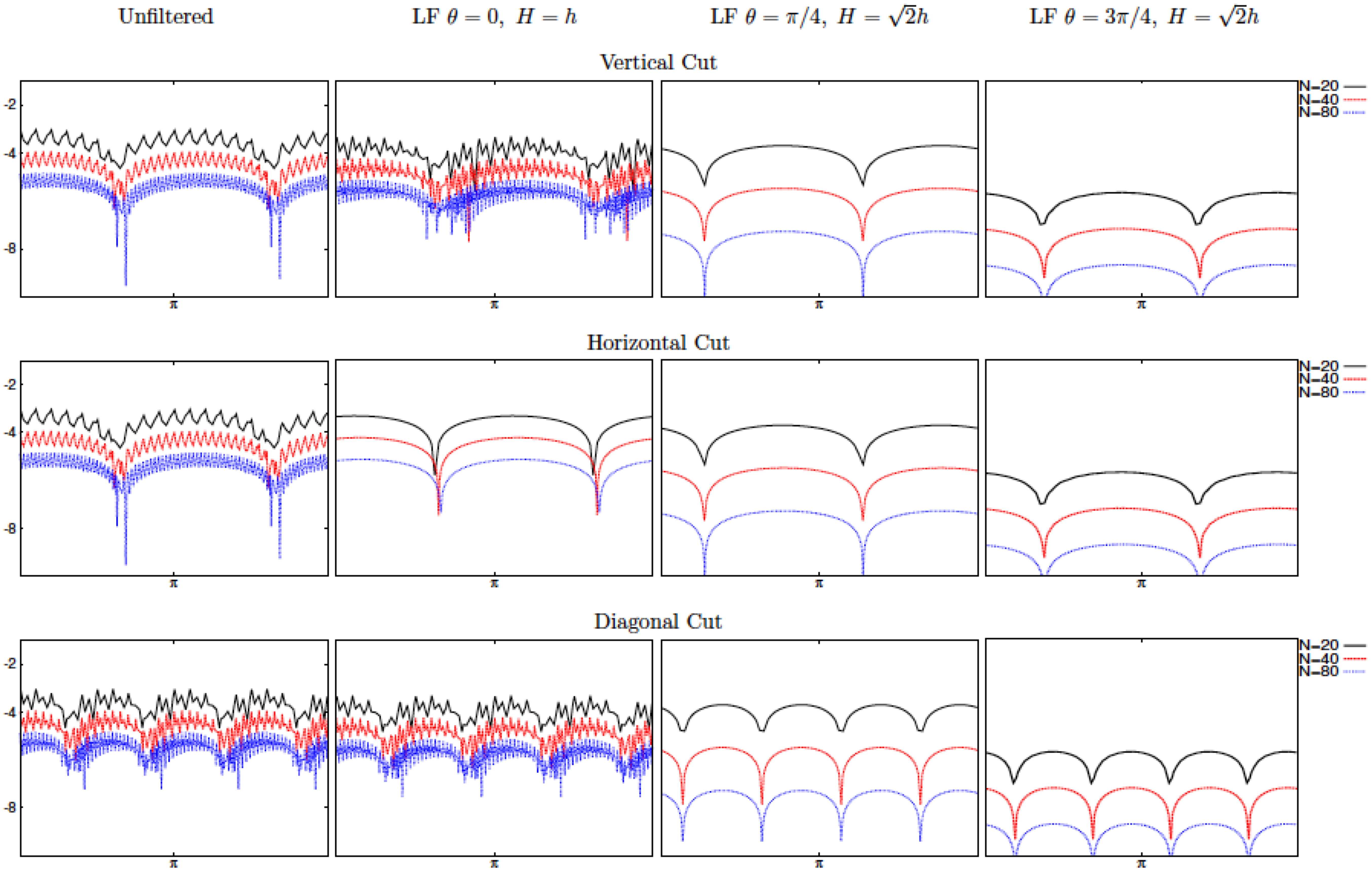


- Numerical test: 2D advection equation, $u(x,u,0)=sin(x+y)$

			$\theta = 3\pi/4$		$\theta = \pi/4$		$\theta = 0$
N	L^2-error	Order	L^2-error	Order	L^2-error	Order	L^2-error
$\mathbb{P}^1$							
20	9.7e-03	-	1.5e-03	-	2.7e-03	-	1.6e-03
40	2.4e-03	2.02	1.9e-04	2.98	2.6e-04	3.33	2.0e-04
80	5.9e-04	2.01	2.4e-05	2.99	2.8e-05	3.21	
$\mathbb{P}^2$							
20	2.4e-04	-	1.5e-06	-	1.4e-04	-	6.1e-06
40	2.9e-05	3.01	4.7e-08	4.99	2.3e-06	5.91	1.2e-07
80	3.6e-06	3.01	1.5e-09	5.00	3.7e-08	5.95	-
$\mathbb{P}^3$							
20	4.5e-06	-	7.7e-10	-	1.6e-05	-	1.4e-07
40	2.8e-07	4.01	6.9e-12	6.79	6.9e-08	7.87	5.6e-10

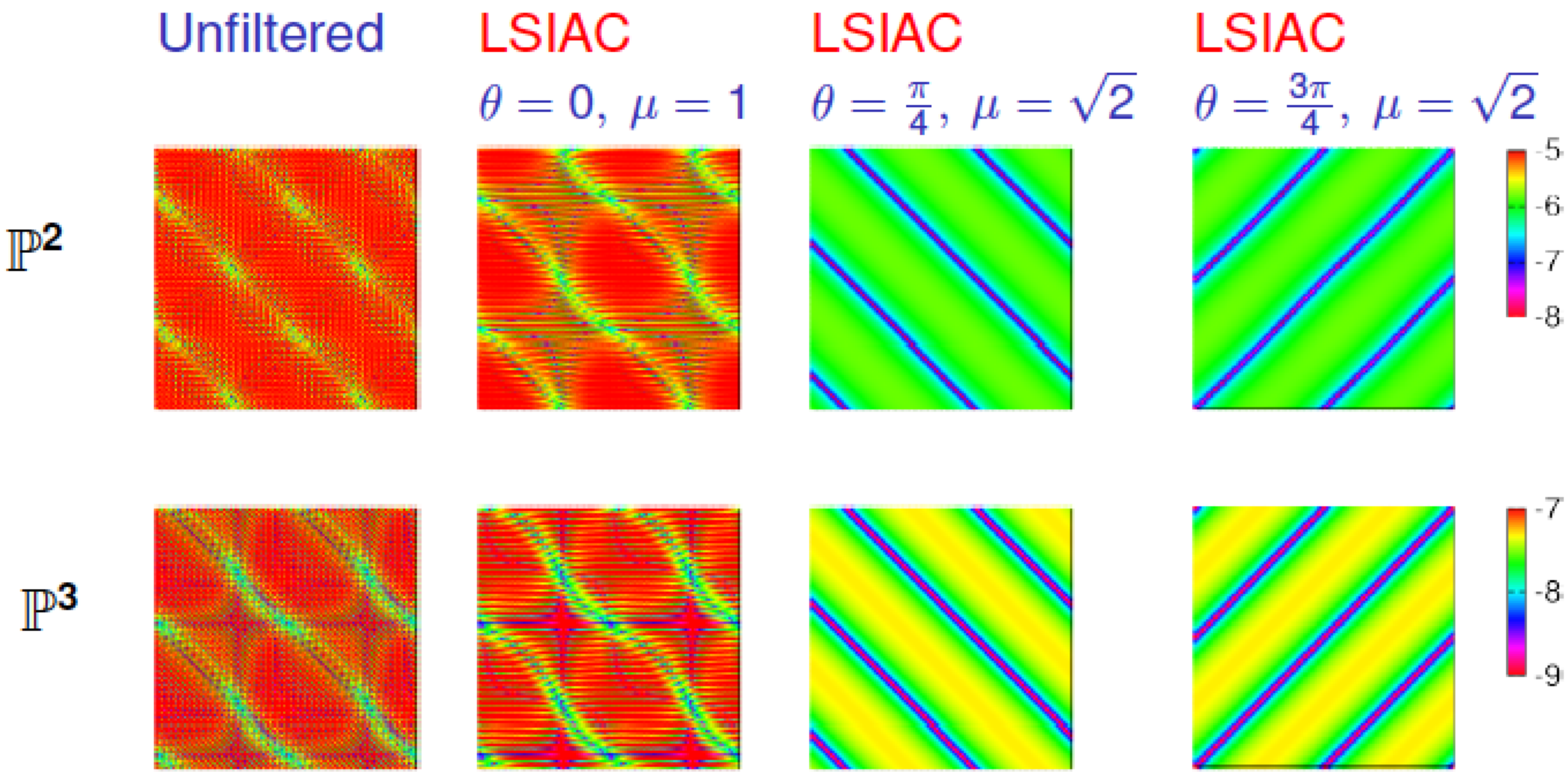
- **Superconvergence** and **error reduction**!

LSIAC FILTER: SMOOTHNESS



LINE SIAC FILTER FOR MULTI-DIMENSIONAL FILTERING

$$u_t + u_x + u_y = 0, \quad u(x, y, 0) = \sin(x)\cos(y)$$



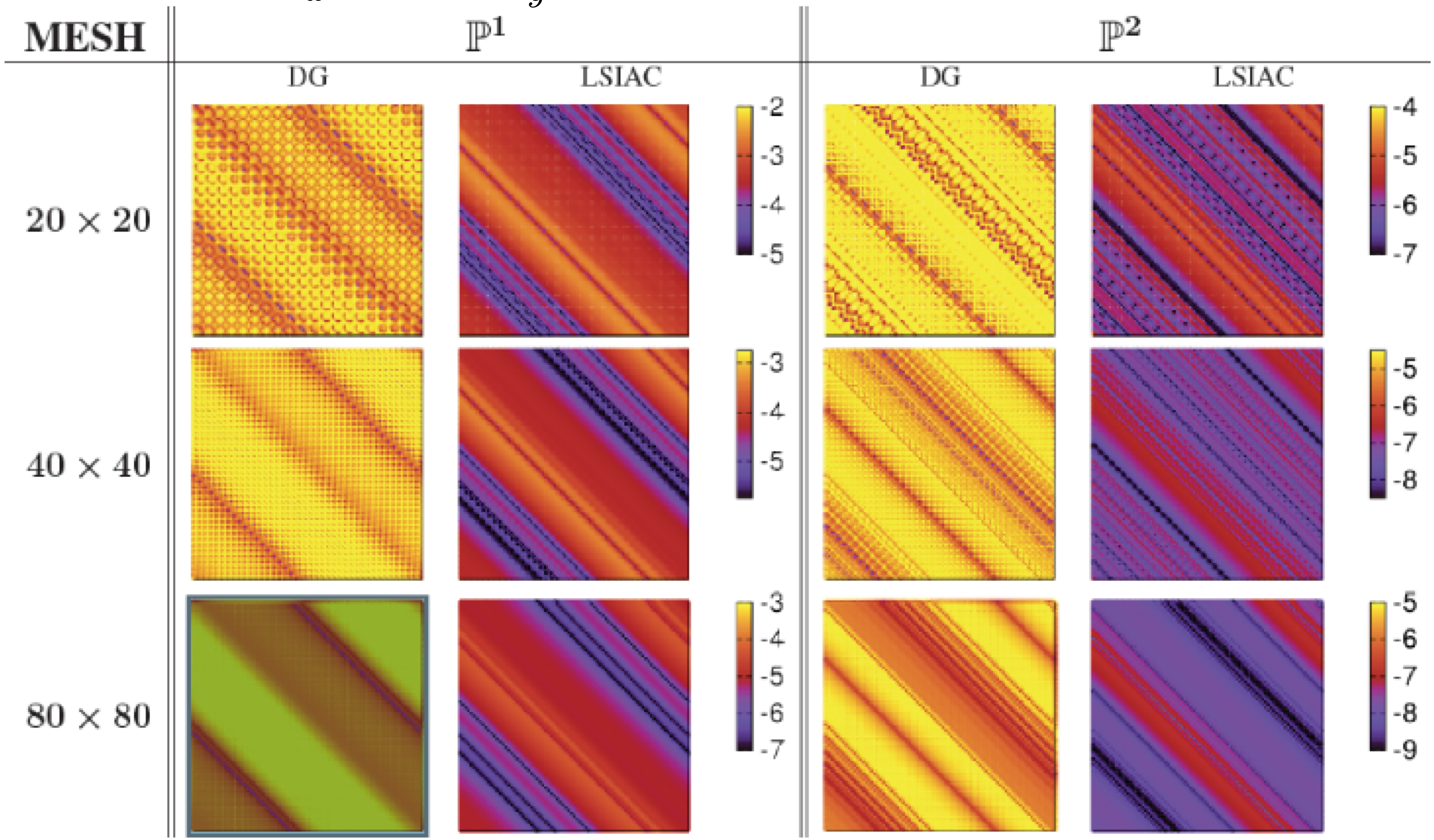
LINE SIAC FILTER FOR MULTI-DIMENSIONAL FILTERING

$$u_t + u_x + u_y = 0, \quad u(x, y, 0) = \sin(x)\cos(y)$$

	Unfiltered		LSIAC					
			$\theta = 0$		$\theta = \pi/4$		$\theta = 3\pi/4$	
N	L^2-Error	Order	L^2-Error	Order	L^2-Error	Order	L^2-Error	Order
				$\mathbb{P}^2$				
20	1.3e-04	-	9.1e-05	-	6.8e-05	-	6.7e-05	-
40	1.6e-05	3.01	1.1e-05	3.02	1.1e-06	5.91	1.1e-06	5.92
80	2.0e-06	3.00	1.4e-06	3.00	1.8e-08	5.95	1.8e-08	5.98
				$\mathbb{P}^3$				
20	2.4e-06	-	1.9e-06	-	8.1-e06	-	8.1e-06	-
40	1.5e-07	4.01	1.1e-07	4.13	3.4e-08	7.87	3.4e-08	7.87
80	9.5e-09	4.00	6.7e-09	4.00	1.4e-10	7.97	1.4e-10	7.97

LINE SIAC FILTER FOR MULTI-DIMENSIONAL FILTERING

$$u_t + \left(\frac{u^2}{2}\right)_x + \left(\frac{u^2}{2}\right)_y = 0, \qquad u(x,y,0) = 2 + \frac{1}{2}\sin(x+y).$$



LINE SIAC FILTER FOR MULTI-DIMENSIONAL FILTERING

$$u_t + \left(\frac{u^2}{2}\right)_x + \left(\frac{u^2}{2}\right)_y = 0, \qquad u(x,y,0) = 2 + \frac{1}{2}\sin(x+y).$$

Nonlinear!

	DG				LSIAC			
N	L^2 error	Order	L^∞ error	Order	L^2 error	Order	L^∞ error	Order
				$\mathbb{P}^1$				
20	6.15e-03		2.82e-02		1.08e-03		2.81e-03	
40	1.51e-03	2.02	7.11e-03	1.99	1.47e-04	2.87	4.26e-04	2.72
80	3.76e-04	2.01	1.78e-03	2.00	1.90e-05	2.96	5.66e-05	2.91
				$\mathbb{P}^2$				
20	1.75e-04		8.09e-04		4.34e-06		1.54e-05	
40	2.02e-05	3.12	1.22e-04	2.73	1.55e-07	4.81	4.31e-07	5.16
80	2.46e-06	3.04	1.67e-05	2.87	5.14e-08	1.59	1.58e-07	1.44

LSIAC FILTER: COMPUTATIONAL COST

- Total operations per point.

Filter Type	Intersection Scans	Integrals	Quadrature Sums
Line Filter	4	10	10
2D Rotated Filter	64	93	8649
2D No Rotation	64	63	3969

- Elapsed time per point.

No. of Splines and degree	Line Filter	2D Rotated Filter	2D No Rotation
$3, 1$	0.09	0.87	0.68
$5, 2$	0.35	3.49	2.60
$7, 3$	0.41	10.42	6.75

SUMMARY

- A **Line SIAC filter** can be applied for multi-dimensional data
  - Reduces error
  - Increases smoothness in all directions
  - Reduced computational cost
  - Improves the convergence rate from $p+1$ to $2p+1$
- Requires choosing the rotation wisely.
- Essential to have the appropriate divided difference estimates.
- Can generalise to *higher dimensions* given the appropriate parameterisation.

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