CutFem and Finite Differences for wave equations

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High order immersed methods

Outline

- Background
- Example in 1D: CutFem & SBP-SAT
- 4th order and beyond

Wave equations

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$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix} u \\ v \end{pmatrix} = A\frac{\partial^2}{\partial x^2}\begin{pmatrix} u \\ v \end{pmatrix} + 2B\frac{\partial^2}{\partial x \partial y}\begin{pmatrix} u \\ v \end{pmatrix} + C\frac{\partial^2}{\partial y^2}\begin{pmatrix} u \\ v \end{pmatrix}$$

Wave equations

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$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2}{\partial t^2}\binom{u}{v} = A\frac{\partial^2}{\partial x^2}\binom{u}{v} + 2B\frac{\partial^2}{\partial x \partial y}\binom{u}{v} + C\frac{\partial^2}{\partial y^2}\binom{u}{v}$$

Wish List:

- High order accuracy
- Regular grid and complex geometry: Immersed boundaries/interfaces
- Explicit time stepping with $k/h \leq C$

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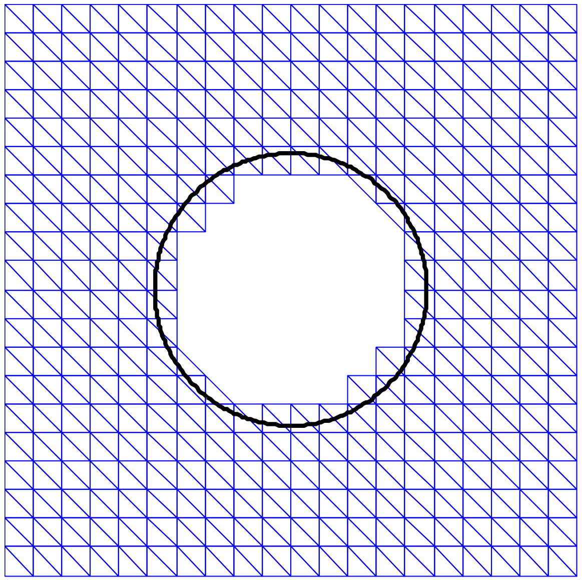
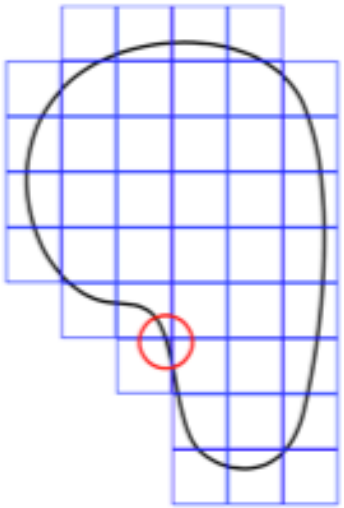
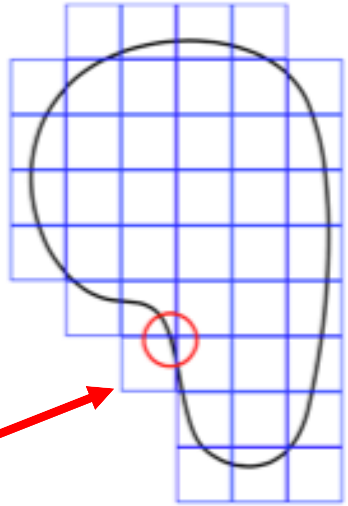
Immersed method?

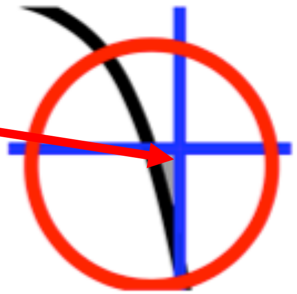




Immersed method

- Almost undetermined DoF?
- Small time step?





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Immersed methods and waves

Finite Difference/Finite Volumes

Much work by Berger, LeVeque, Shu, Petersson, Lombard, Ditkowski, Tsynkov,…(many more!)

FD/FV: no general technique for provably stable high order

Immersed methods and waves

Finite Element methods

CutFem (Burman,Hansbo,Larson…),

Xfem, etc

Mainly developed for elliptic & parabolic

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Wave equations?

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Cut FEM for wave equations

CutFem for wave equation and elastic wave equations

Sticko,GK2016, Sticko,GK2017, Sticko,Ludvigsson,GK2018

- Provably stable, immersed, order 2&4,
- Explicit, with time step $k \sim h$ independent of cuts

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Cut FEM for wave equations

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- Provably stable, immersed, order 2&4,
- Explicit, with time step $k \sim h$ independent of cuts

Compare with FD, SBP-SAT in particular?

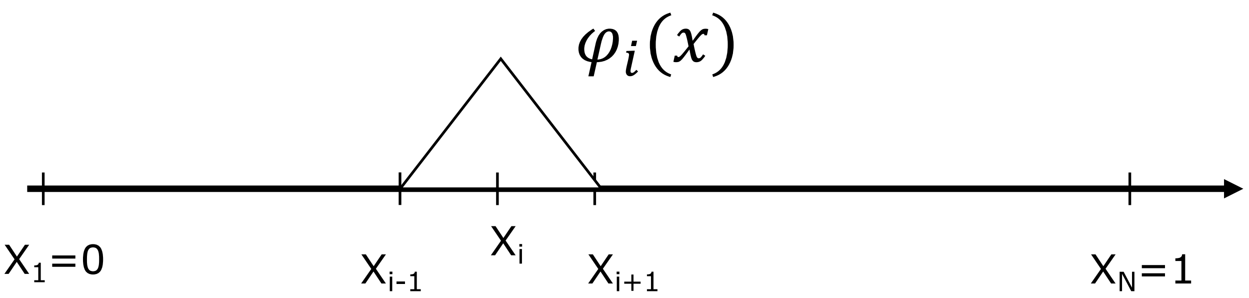
Order 6,8…?

FEM and FD, conforming

$$u_{tt} = u_{xx}, \qquad 0 \leq x \leq 1, t \geq 0$$
$$u(0,t) = u(1,t) = 0$$

FEM, piecewise linear

$$u_{tt} = u_{xx}, \qquad 0 \le x \le 1, t \ge 0$$
$$u(0,t) = u(1,t) = 0$$



$$u^h(x,t) = \sum_{i=1}^{N} u_i(t)\varphi_i(x)$$

FEM with weak boundary condition

$$u_{tt} = u_{xx}, \qquad 0 \leq x \leq 1, t \geq 0$$
$$u(0,t) = u(1,t) = 0$$

$$(u_{tt}^h, v^h) = -(u_x^h, v_x^h) + [u_x^h v^h] + [u^h v_x^h] - \frac{\gamma}{h}[u^h v^h]$$
$$\Leftrightarrow$$
$$MU_{tt} = \left(-A + D_B + D_B^T - \frac{\gamma}{h}E\right)U$$

$$M_{ij} = \int \varphi_i \varphi_j dx$$

$$A_{ij} = \int \varphi_{ix} \varphi_{jx} dx$$

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$$\Leftrightarrow$$
$$MU_{tt} = \left(-A + D_B + D_B^T - \frac{\gamma}{h}E\right)U$$

$$M = h\begin{pmatrix} 0.5 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 0.5 \end{pmatrix}$$

Trapezoidal rule!

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FEM with weak boundary condition

$$MU_{tt} = QU, \qquad Q = -A + D_B + D_B^T - \frac{\gamma}{h} E$$

$$A = \frac{1}{h}\begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{pmatrix}$$

$$D_B = \frac{1}{h}\begin{pmatrix} 1 & -1 & & & \\ 0 & 0 & 0 & & \\ & & \ddots & & \\ & & 0 & 0 & 0 \\ & & & -1 & 1 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & 1 \end{pmatrix}$$

- Q is symmetric and negative definit if $\gamma > 1$ → stability!
- FEM with weak BC ⟷ SBP+SAT
- Can extend to multi-D

Explicit time stepping: $U_{tt} = M^{-1} Q U$
Time step restriction?

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Stability region must include *[-ic,ic].*

$$k\sqrt{\lambda_{\infty}(-M^{-1}Q)} \leq c$$

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Explicit time stepping: $U_{tt} = M^{-1}Q\,U$ Time step restriction?

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$$\lambda_\infty(-M^{-1}Q) \sim h^{-2}$$

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Explicit time stepping: $U_{tt} = M^{-1}Q\,U$ Time step restriction?

Stability region must include *[-ic,ic].*

$$k\sqrt{\lambda_\infty(-M^{-1}Q)} \le c$$

$$\lambda_\infty(-M^{-1}Q) \sim h^{-2}$$

$$\frac{k}{h} \le C$$

CutFEM,

$$u_{tt} = u_{xx}, \qquad 0 \le x \le 1 + \delta h, t \ge 0$$
$$u(0,t) = u(1 + \delta h, t) = 0$$

Same weak form: integrate only part of *$[x_{N-1}, x_N]$*

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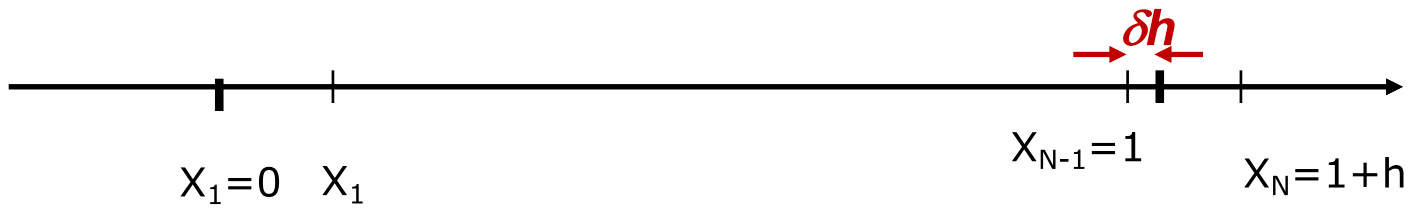

Same weak form: integrate only part of $[x_{N-1}, x_N]$

CutFEM

$$u_{tt} = u_{xx}, \qquad 0 \le x \le 1 + \delta h, t \ge 0$$
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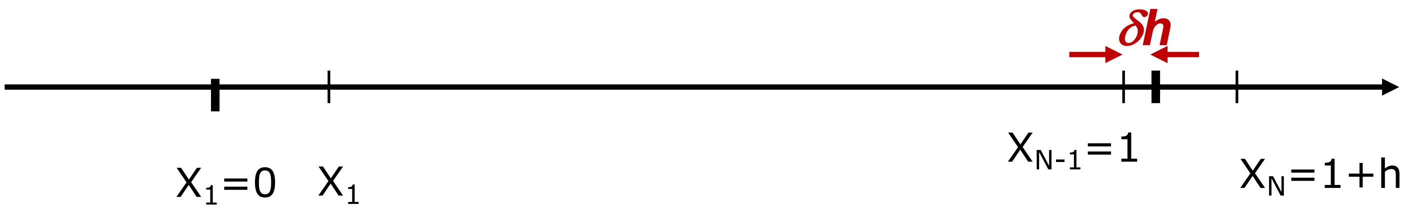

Same weak form: integrate only part of *[x*$[x_{N-1}, x_N]$

$$\underbrace{\begin{pmatrix} M_{N-1,N-1} & M_{N-1,N} \\ M_{N,N-1} & M_{N,N} \end{pmatrix}}_{\widetilde{M}} = \frac{h}{2}\begin{pmatrix} 2+\delta(1-\delta^2) & \delta^2(1-\delta) \\ \delta^2(1-\delta) & 2\delta^3 \end{pmatrix}$$

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CutFem = SBP?

$$\tilde{A} = \begin{pmatrix} 1+\delta & -\delta \\ -\delta & \delta \end{pmatrix}$$

$$\widetilde{D}_B = \begin{pmatrix} -1+\delta & 1-\delta \\ -\delta & \delta \end{pmatrix}$$

$$\tilde{E} = \begin{pmatrix} (1-\delta)^2 & \delta(1-\delta) \\ \delta(1-\delta) & \delta^2 \end{pmatrix}$$

Interpret as SBP finite difference method!

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CutFem = SBP?

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Problem when $\delta \ll 1$: M almost singular

$$k \ll h$$

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Problem when $\delta \ll 1$: M almost singular

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Stabilize!

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CutFEM

$$u_{tt} = u_{xx}, \qquad 0 \leq x \leq 1 + \delta h, t \geq 0$$
$$u(0,t) = u(1 + \delta h, t) = 0$$

Jump stabilization: add to weak form

$$-\gamma_s h [u_x^h]_{1^-}^{1^+} [v_x^h]_{1^-}^{1^+}$$

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CutFEM

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Jump stabilization: add to weak form

$$-\gamma_s h[u_x^h]_{1^-}^{1^+}[v_x^h]_{1^-}^{1^+}$$

Penalize jumps in normal derivative

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CutFEM

$$u_{tt} = u_{xx}, \qquad 0 \le x \le 1 + \delta h, t \ge 0$$
$$u(0,t) = u(1+\delta h, t) = 0$$

Jump stabilization: add to lower right of *M* and *Q*

$$\gamma_s \begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix} = \gamma_s \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 \end{pmatrix}$$

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CutFEM

$$u_{tt} = u_{xx}, \qquad 0 \le x \le 1 + \delta h, t \ge 0$$
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Stability by ”fem machinery”, *k* independent of δ, multi-D.

Second order immersed SBP-SAT method!

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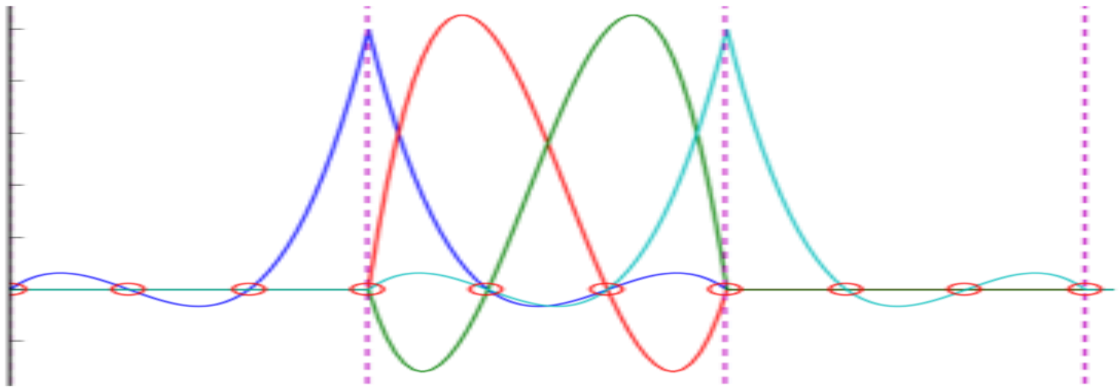
Higher order

1) Standard FEM ⟷ FD with alternating stencils

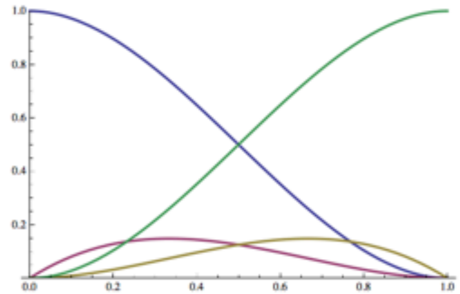


- Equivalent to a *strange* SBP-SAT
- Time step $k/h \leq c_p$ decreses with p
- Condition number $\kappa(M)$ grows with p

Higher order

2) "Hermite" FEM $\longleftrightarrow$ compact FD: find u & u_x

$$\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{bmatrix} := \begin{bmatrix} 1-3x^2+2x^3 \\ x-2x^2+x^3 \\ x^2-x^3 \\ 3x^2-2x^3 \end{bmatrix}$$



$$\underbrace{\begin{pmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{pmatrix}}_{M_4} \begin{pmatrix} u \\ v \end{pmatrix}_{tt} = -\underbrace{\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix}}_{Q_4} \begin{pmatrix} u \\ v \end{pmatrix}$$

M_4* and *Q_4* consist of *D_0*, *D_+D_-* and *I

Time step restriction: $\frac{k}{h} < C_p$, decreases slower with p ?

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Cut Hermite FEM

$$\underbrace{\begin{pmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{pmatrix}}_{M_4} \begin{pmatrix} u \\ v \end{pmatrix}_{tt} = \underbrace{-\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}}_{Q_4}$$

Lower right corners of blocks are modified for accuracy:

$$\widetilde{M}_{ij}(\delta), \qquad \tilde{Q}_{ij}(\delta)$$

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Cut Hermite FEM

$$\underbrace{\begin{pmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{pmatrix}}_{M_4} \begin{pmatrix} u \\ v \end{pmatrix}_{tt} = -\underbrace{\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix}}_{Q_4} \begin{pmatrix} u \\ v \end{pmatrix}$$

Lower right corners of blocks are modified for accuracy:

$$\widetilde{M}_{ij}(\delta), \qquad \tilde{Q}_{ij}(\delta)$$

Need CutFem stabilization for stability, time step & $\kappa(M)$

$$-\gamma_s^{(2)} h[u_{xx}^h][v_{xx}^h]_{x=1} - \gamma_s^{(3)} h[u_{xxx}^h][v_{xxx}^h]_{x=1}$$

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Cut Hermite FEM

$$\underbrace{\begin{pmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{pmatrix}}_{M_4} \begin{pmatrix} u \\ q \end{pmatrix} = -\underbrace{\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix}}_{Q_4}$$

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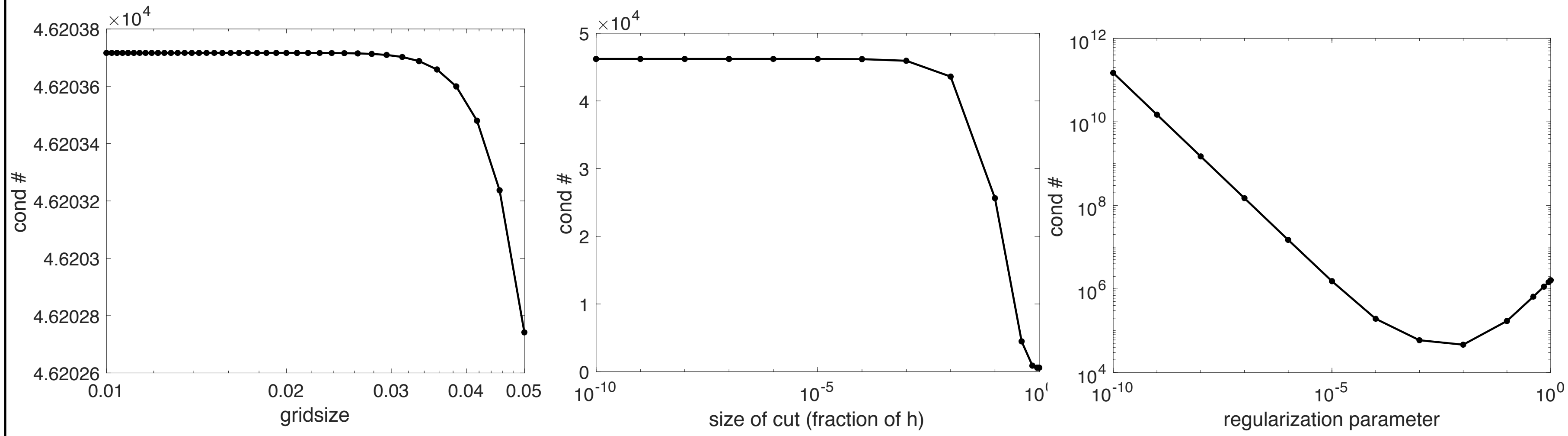
Need CutFem stabilization for stability, time step & $\kappa(M)$

Add to M_{ij} and Q_{ij} lower right corner:

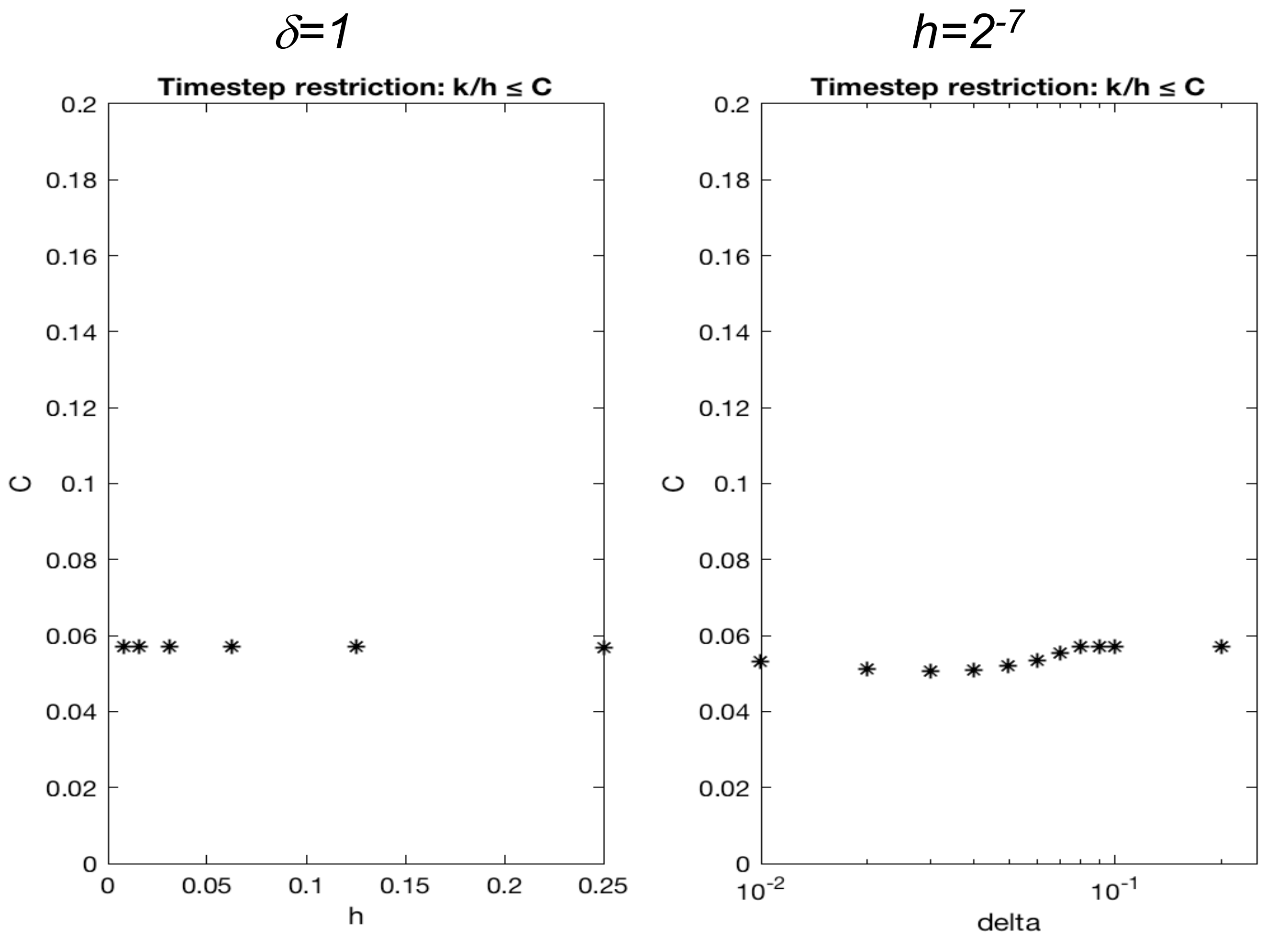
$$h\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix} \text{ and } \frac{1}{h}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

Cut Hermite FEM: Mass matrix condition number



- Mass matrix conditioning is independent of h and δ.
- Robust with respect to parameter

Cut Hermite FEM: time step



Time-step restriction is independent of δ and h!

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Summary

- The second order CutFem method is an immersed SBP-SAT Finite Difference method
- The Cut Hermite Finite Element Method is stable, explicit time steps independent of cut,
- Is Cut Hermite Finite Element Method an immersed 4th order SBP-SAT Finite Difference method of compact type?
- Time step restriction when order increases?

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