A relationship between the Shock-Capturing and Vorticity Confinement methods

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Outline

Introduction

- Vorticiy-confinement methods - developed initially for incompressible flow, enhance resolution of vortical structures.
- Shock-capturing methods - for computing compressible flow with shock waves.
- There appears to exist a (surprising ?) commonality between the two methods.
- Exploration of this commonality leads to devising a unified approach.

Vorticity confinement methods

- Developed by John Steinhoff for incompressible flow equations
- There exist two approaches: VC1 (early 90's) and VC2 (late 90's)
- Both are concerned with an addition of a nonlinear mechanism to a numerical scheme
- VC2 is relevant for the purpose of this work

Incompessible NS equations

Continuity equation and momentum equations

$$\nabla \cdot \mathbf{v} = 0$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = \mu \nabla^2 \mathbf{v}$$

with p - pressure, $\mathbf{v}$ - velocity vector, μ - viscosity coefficient.
Using the identity

$$\nabla^2 \mathbf{v} = \nabla \left(\nabla \cdot \mathbf{v} \right) - \nabla \times \nabla \times \mathbf{v}$$

the momentum equations can be recast

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = -\mu \nabla \times \boldsymbol{\omega}$$

VC2 method

The "anti-diffusion" term

$$\mathbf{s} = \nabla \times \boldsymbol{\varpi}$$

Together with the dissipation can be recast as

$$\mu \nabla^2 \mathbf{v} - \varepsilon \mathbf{s} = \nabla \times (\mu \boldsymbol{\omega} - \varepsilon \boldsymbol{\varpi})$$

where

$$\boldsymbol{\varpi} = \frac{\boldsymbol{\omega}}{\bar{\omega}} \left[\frac{\sum_l (\bar{\omega}_l)^{-1}}{N} \right]^{-1}$$

with

$$\bar{\omega} = \|\boldsymbol{\omega}_l\| + \delta$$

Compressible flow applications

- A difficulty: both VC and Shock Capturing involve artificial non-linearity
  - Most of the VC compressible flow applications are subsonic
  - Work by Hu (2001): VC on top of the FCT method - supersonic flow tests.

Preliminary remarks

The dissipation & VC2 term in tensorial form

$$\nabla \times (\mu \boldsymbol{\omega} - \varepsilon \boldsymbol{\varpi}) \equiv \nabla \cdot \left[\mu \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \right.$$
$$\left. - \varepsilon \begin{pmatrix} 0 & -\varpi_3 & \varpi_2 \\ \varpi_3 & 0 & -\varpi_1 \\ -\varpi_2 & \varpi_1 & 0 \end{pmatrix} \right]$$

Euler equations for compressible flow

The Euler system of equations for compressible flow in the conservation form

$$\mathbf{u}_t + [\mathbf{F}(\mathbf{u})]_x + [\mathbf{G}(\mathbf{u})]_y + [\mathbf{H}(\mathbf{u})]_z = 0$$

where the vector of unknowns and the x-flux:

$$\mathbf{u} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{pmatrix} \qquad \mathbf{F}(\mathbf{u}) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ \rho uH \end{pmatrix}$$

$\mathbf{v} = (u, v, w)$ - the velocity vector, ρ - the density, p - the pressure, E - the total specific energy and $H = E + p/\rho$ - specific enthalpy. The ideal gas equation of state $p = (\gamma - 1)\rho E$, where γ is the specific heats ratio.

The numerical scheme

The upwind scheme's numerical flux

$$\hat{\mathbf{F}}^U_{i+1/2} = \frac{1}{2}\left[\mathbf{F}(\mathbf{u}_i) + \mathbf{F}(\mathbf{u}_{i+1})\right] - \underbrace{\frac{1}{2}\left(R\,|\Lambda|\right) Q_{i+1/2}(\mathbf{q})}$$

Jacobian at the cell face $i + 1/2$

$$A = \mathbf{F}'_{\mathbf{u}}|_{x=x_{i+1/2}}$$

based either upon Roe-averaging procedure, $\mathbf{q}$ is a vector of characteristic variables,

$$Q_{i+1/2}(\mathbf{q}) = \delta_{i+1/2}(\mathbf{q}) \equiv \mathbf{q}_{i+1} - \mathbf{q}_i$$

with $\delta_{i+1/2}(\ldots)$ denoting undivided difference, R - is the matrix of the Jacobians' right eigenvectors.
Eigenvalues of A: three identical - u, "advective"; two others - $(u \pm c)$ - "acoustic".

Primitive variables formulation

$$
\begin{aligned}
\frac{\partial s}{\partial t} + \mathbf{v} \cdot \nabla s &= 0 \\
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{1}{\rho} \nabla p &= 0 \\
\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \rho c^2 \nabla \cdot \mathbf{v} &= 0
\end{aligned}
$$

where s is the entropy and c - the speed of sound $c^2 = \gamma p/\rho$.
The transformation matrix between the primitive and conservative variables

$$
T = \begin{pmatrix}
1 & 0 & 0 & 0 & 1/c^2 \\
u & \rho & 0 & 0 & u/c^2 \\
v & 0 & \rho & 0 & v/c^2 \\
w & 0 & 0 & \rho & w/c^2 \\
\mathbf{v}^2 & \rho u & \rho v & \rho w & \mathbf{v}^2/c^2
\end{pmatrix}
$$

Relevant artificial viscosity terms

Formulate the dissipative portion of the upwind scheme numerical flux for primitive variables equations.
Single out the following terms

$$\frac{h}{2}\begin{pmatrix} 0 & |v|\,u_y & |w|\,u_z \\ |u|\,v_x & 0 & |w|\,v_z \\ |u|\,w_x & |v|\,w_y & 0 \end{pmatrix} = \frac{h}{2}\begin{pmatrix} 0 & u_y & u_z \\ v_x & 0 & v_z \\ w_x & w_y & 0 \end{pmatrix}\mathcal{M} \qquad (1)$$

with

$$\mathcal{M} = \begin{pmatrix} |u| & 0 & 0 \\ 0 & |v| & 0 \\ 0 & 0 & |w| \end{pmatrix}$$

Some details of the TVD approach

The corresponding components of vector $Q_{i+1/2}(\mathbf{q})$

$$Q_{i+1/2}(v) = \delta_{i+1/2}(v), \quad Q_{i+1/2}(w) = \delta_{i+1/2}(w)$$

A second order upwind scheme - redefining v component (assuming $u > 0$)

$$Q_{i+1/2}(v) = \delta_{i+1/2}(v) - \delta_{i-1/2}(v)$$

A TVD-type scheme, again, by redefining

$$Q_{i+1/2}(v) = \delta_{i+1/2}(v) - \delta_{i-1/2}(v)\,\phi\left[r^{+}_{i+1/2}(v)\right]$$

where

$$r^{+}_{i+1/2}(v) = \frac{\delta_{i+1/2}(v)}{\delta_{i-1/2}(v)}$$

and $\phi(r)$ is one of the so-called limiter-functions.

"Augmenting" the artificial dissipation

Augment the singled out terms by subtracting the transposed tensor

$$\frac{h}{2}\left[\begin{pmatrix} 0 & u_y & u_z \\ v_x & 0 & v_z \\ w_x & w_y & 0 \end{pmatrix} - \begin{pmatrix} 0 & u_y & u_z \\ v_x & 0 & v_z \\ w_x & w_y & 0 \end{pmatrix}^T\right]\mathcal{M}$$

$$= \frac{h}{2}\begin{pmatrix} 0 & -\omega^z & \omega^y \\ \omega^z & 0 & -\omega^x \\ -\omega^y & \omega^x & 0 \end{pmatrix}\mathcal{M}$$

- a skew-symmetric form

“Augmenting” the artificial dissipation (cont-d)

In addition to the “regular” undivided differences, introduce the transverse ones: in y-direction

$$\tau^{y}_{i+1/2}(u) = \frac{\left(u_{i+1,j+1,k} - u_{i+1,j-1,k}\right) + \left(u_{i,j+1,k} - u_{i,j-1,k}\right)}{4}$$

and in z-direction

$$\tau^{z}_{i+1/2}(u) = \frac{\left(u_{i+1,j,k+1} - u_{i+1,j,k-1}\right) + \left(u_{i,j,k+1} - u_{i,j,k-1}\right)}{4}$$

Introduce the “undivided vorticity” components:
$\underline{\omega}^{z} \approx h\omega^{z}$; $\quad \underline{\omega}^{y} \approx h\omega^{y}$ which are evaluated as follows

$$\underline{\omega}^{z}_{i+1/2} \equiv \left[\delta_{i+1/2}(v) - \tau^{y}_{i+1/2}(u)\right]$$

$$\underline{\omega}^{y}_{i+1/2} \equiv \left[\delta_{i+1/2}(w) - \tau^{z}_{i+1/2}(u)\right]$$

A first-order upwind scheme flux with augmented (or “vorticity”) artificial dissipation is defined by:

$$Q_{i+1/2}\left(v\right) = \underline{\omega}^{z}_{i+1/2}, \qquad Q_{i+1/2}\left(w\right) = \underline{\omega}^{y}_{i+1/2}$$

TVD-VC scheme formulation

The next step is to devise higher order corrections.
By analogy to the "regular" TVD scheme (assuming $u > 0$):

$$Q_{i+1/2}(v) = \underline{\omega}^z_{i+1/2} - \underline{\omega}^z_{i-1/2}\phi\left(R^+_{i+1/2}\right) \equiv \underline{\omega}^z_{i+1/2} - \varpi^z_{i-1/2}$$

where

$$R^+_{i+1/2} = \frac{\underline{\omega}^z_{i+1/2}}{\underline{\omega}^z_{i-1/2}}$$

The component $Q_{i+1/2}(w)$ is evaluated in analogous manner, as well the other elements of the "limited vorticity" correction tensor

- The entropy and the "acoustic" characteristic variables are treated in the standard way.

TVD-VC scheme formulation (cont-d)

The entire 2nd order upwind "vorticity" dissipation is

$$\frac{1}{2}\left[\begin{pmatrix} 0 & -\underline{\omega}^z & \underline{\omega}^y \\ \underline{\omega}^z & 0 & -\underline{\omega}^x \\ -\underline{\omega}^y & \underline{\omega}^x & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\varpi^z & \varpi^y \\ \varpi^z & 0 & -\varpi^x \\ -\varpi^y & \varpi^x & 0 \end{pmatrix}\right]\mathcal{M}$$

where ϖ^α terms are the "limited" vorticity components (second order corrections)
- resemblance to the VC2 scheme !

Some remarks

- Upwind TVD-VC scheme - resembles the VC2 method, though the key differences are
  - limiting based on vorticity components (not on the vorticity vector magnitude)
  - limiting along a grid-line (not a more general neighborhood)
- The general strategy for SC-VC
  - single out the relevant velocity error components
  - augment them so that they are expressed via vorticity components.
  - contstruct (limited) correction, also formulated based upon vorticity components.
- Flux-splitting TVD - straightforward.
- Reasonably easy to retrofit existing codes.

Isentropic vortex example

(Studied by Shu and Yee)

Computational domain $\Omega = \{[-5, 5] \times [-5, 5]\}$:

$$
\begin{aligned}
T &= 1 - \frac{\gamma - 1}{8\gamma\pi} \exp\left(1 - r^2\right) \\
p &= T^{\frac{1}{\gamma - 1}} \\
\rho &= \rho T \equiv \rho^{\gamma}
\end{aligned}
$$

Velocities

$$
(u, v) = \frac{\epsilon}{2\pi} \exp\left(\frac{1 - r^2}{2}\right)(-y, x)
$$

with $r = \sqrt{x^2 + y^2}$, $\epsilon = 5$.

Computational grid: 70×70 *cells.*

Isentropic vortex testcase 1

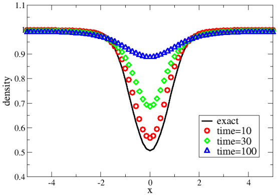

Figure: TVD-VC

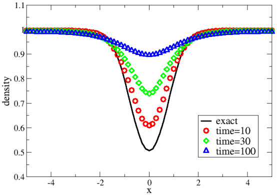

Figure: TVD

Sweby limiter, $\beta = 1$ (identical to minmod).

Isentropic vortex testcase 2

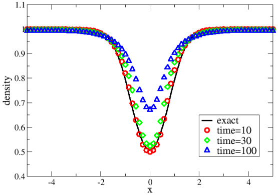

Figure: TVD-VC

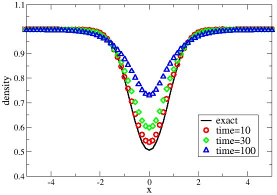

Figure: TVD

Sweby limiter, $\beta = 1.1$.

Accuracy verification

mesh-size	L_1 error	L_1 order	L_∞ error	L_∞ order
1/10	$4.1362E-04$		$8.3428E-03$	
1/20	$1.3635E-04$	1.60	$3.6567E-03$	1.18
1/40	$3.7331E-05$	1.87	$1.0919E-03$	1.74
1/80	$9.8754E-06$	1.92	$2.6601E-04$	2.04

Table: Accuracy test for TVD-VC method, Sweby limiter with $\beta = 1.1$, time $t = 2$. Errors in ρ are presented.

Shock-capturing properties verification

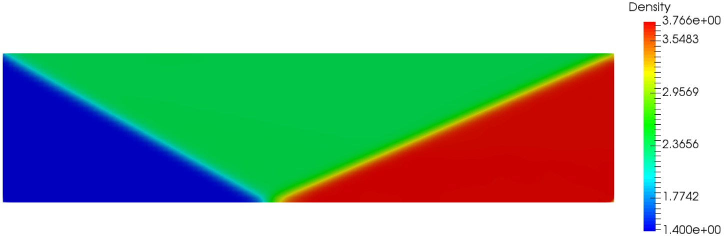

Figure: TVD-VC, shock reflection from a wall, computational grid with mesh-size $\Delta x = 1/30$.

WENO-VC scheme formulation principles

- Reformulate a chosen WENO scheme using undivided differences (like the original FD ENO methods by Shu&Osher 1988)
- Apply the previously formulated strategy
  - single out the relevant (velocity) error components
  - augment them so that they are expressed via vorticity components
  - construct limited corrections (vorticity based)

The basic method's choice

- Choice - 5th order WENO scheme (Shu 2003)
- Conversion to WENO-VC - following the previously formulated strategy
- As an illustration - the vorticity base smoothness monitor

$$
\begin{aligned}
\beta_1 &= \frac{13}{12}\left(-\underline{\omega}^z_{i-3/2}+\underline{\omega}^z_{i-1/2}\right)^2+\frac{1}{4}\left(-\underline{\omega}^z_{i-3/2}+3\underline{\omega}^z_{i-1/2}\right)^2\\
\beta_2 &= \frac{13}{12}\left(-\underline{\omega}^z_{i-1/2}+\underline{\omega}^z_{i+1/2}\right)^2+\frac{1}{4}\left(-\underline{\omega}^z_{i-1/2}-\underline{\omega}^z_{i+1/2}\right)^2\\
\beta_3 &= \frac{13}{12}\left(-\underline{\omega}^z_{i+1/2}+\underline{\omega}^z_{i+3/2}\right)^2+\frac{1}{4}\left(-\underline{\omega}^z_{i+1/2}+3\underline{\omega}^z_{i+3/2}\right)^2
\end{aligned}
$$

Isentropic vortex example

W

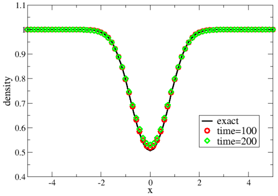

Figure: WENO

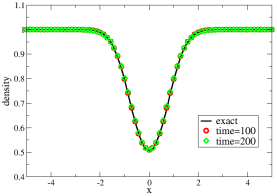

Figure: WENO-VC

Accuracy verification

mesh-size	L_1 error	L_1 order	L_∞ error	L_∞ order
$1/10$	$5.0987E-06$		$1.1228E-04$	
$1/20$	$2.0331E-07$	4.65	$3.3317E-06$	5.07
$1/40$	$7.5689E-09$	4.75	$1.4198E-07$	4.55
$1/80$	$2.3976E-10$	4.98	$3.5008E-09$	5.34

Table: WENO-VC method, isentropic vortex problem, time $t = 2$. Different norm of error in density ρ are presented.

Rayleigh-Taylor instability

time $t = 1.95$

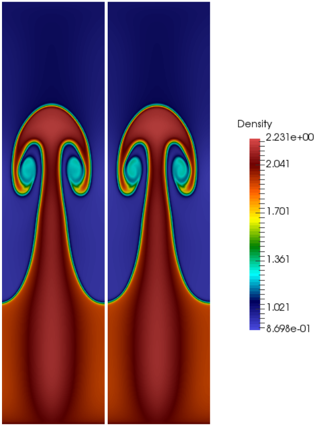

Figure: $\Delta x = 1/240$, WENO-VC - left, WENO -right.

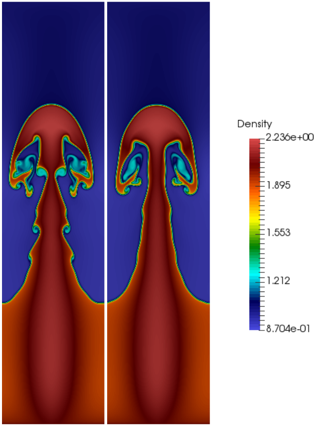

Figure: $\Delta x = 1/480$, WENO-VC left, WENO - right.

Double Mach reflection

$\Delta x = 1/256$, time $t = 2$

WENO-VC

WENO

Double Mach reflection

$\Delta x = 1/512$, time $t = 2$

WENO-VC

WENO

Double Mach reflection

$\Delta x = 1/512$, time $t = 2$

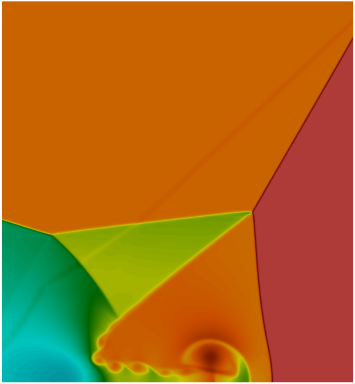

Figure: WENO-VC

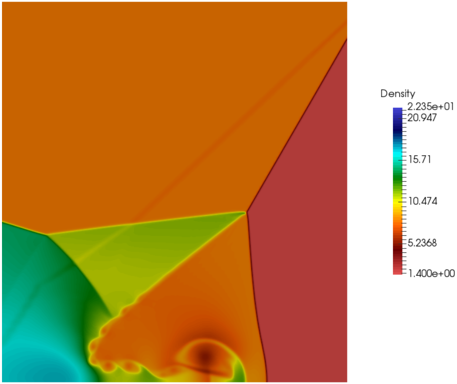

Figure: WENO

Double Mach reflection (vorticity)

$\Delta x = 1/512$, time $t = 2$

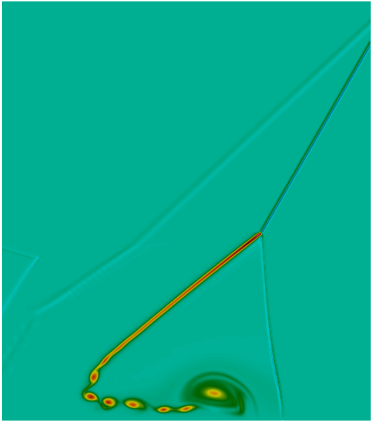

Figure: WENO-VC

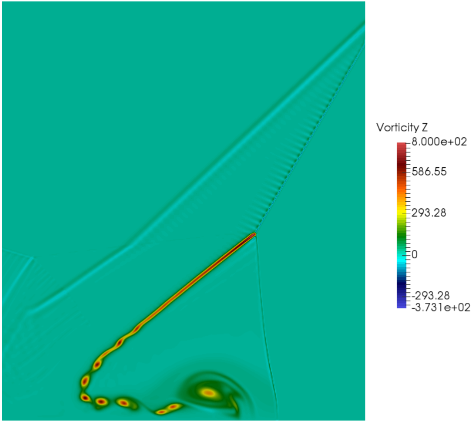

Figure: WENO

Double Mach reflection (vorticity)

$\Delta x = 1/256$

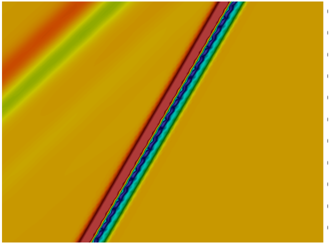

Figure: WENO-VC

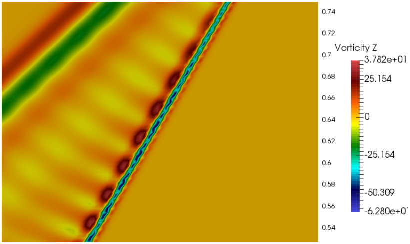

Figure: WENO

Double Mach reflection (vorticity)

$\Delta x = 1/512$

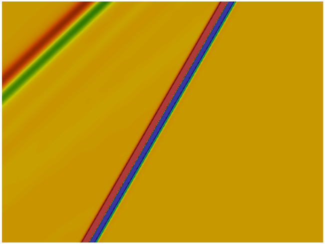

Figure: WENO-VC

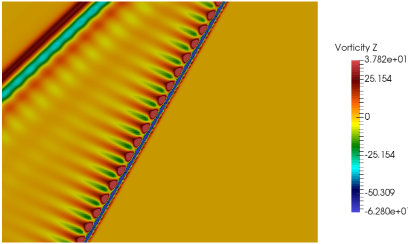

Figure: WENO

Conclusions and future work

- A certain unification of Vorticity Confinement and Shock Capturing methods proposed.
- It constitutes a certain departure from the dimension-by-dimension approach, since the multidmensional quantities (vorticity) are involved.
- The numerical results demonstrate certain advantages of the new approach
  - improved resolution of vortical flows
  - elimination of a certain numerical artifact
- The future plans:
  - factorizable Shock-Capturing higher order methods

Thank you for your attention !