Optimal Truncation of Unbounded Anisotropic Elastic Computational Domains

Dan Givoli
Dept. of Aerospace Engineering
Technion – Israel Institute of Technology

Collaborators: **Tom Hagstrom (SMU), Jacobo Bielak (CMU), Daniel Rabinovich (Technion), Shmuel Vigder (IEC)**

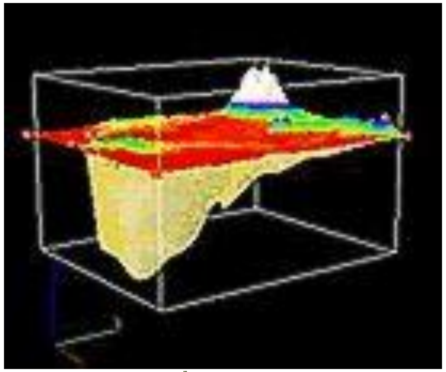
Outline:

- Waves in anisotropic media, “inverse modes”
- Stability of Absorbing Boundary Conditions (ABCs)
- The Energy-Rate Reflection Coefficient (ERRC)
- An optimal ABC & results
- Double Absorbing Boundary (DAB) formulations

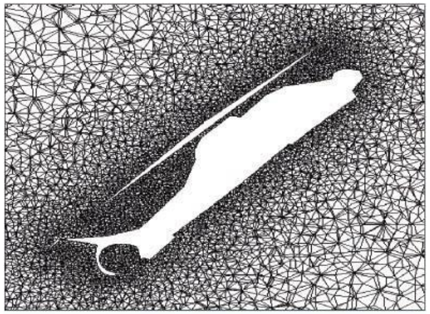
Waves in Unbounded Media

Applications:

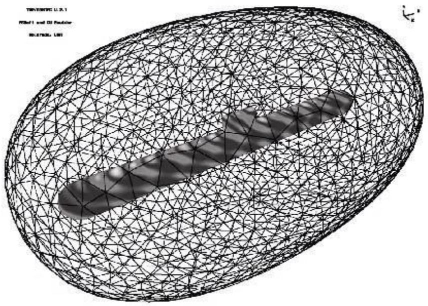
- Underwater acoustics
- Geophysics
- Electromagnetics
- Aerodynamics
- Oceanography
- Meteorology
- and more ……..



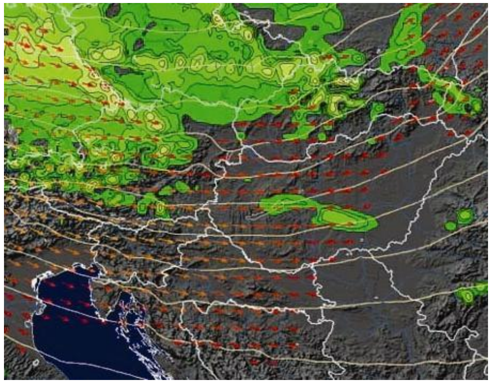
J. Tromp, CalTech



Finite element model of the flow/waves around a rotorcraft (Tayfun E. Tezduyar and Brian Matheis)

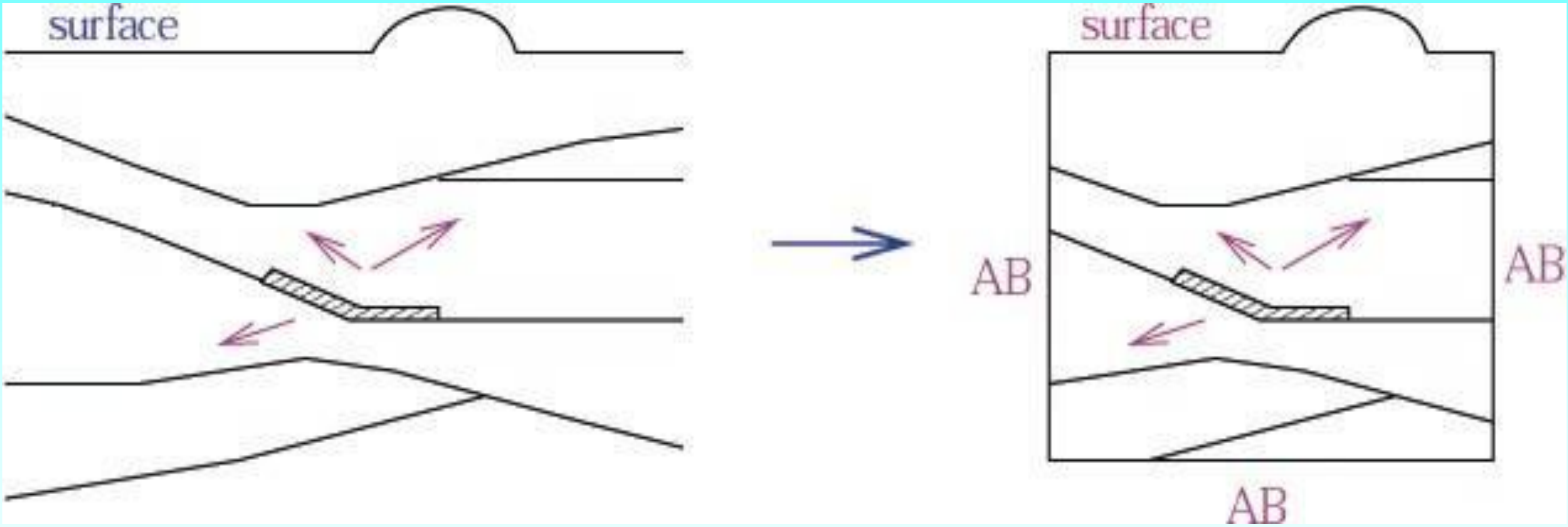


C. Farhat *et al*.

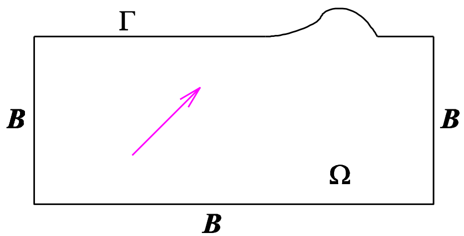


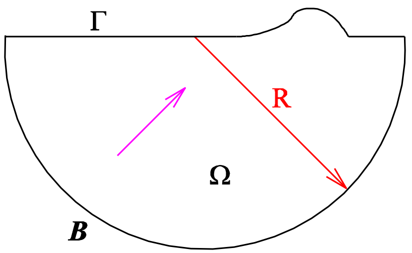
Hungarian Meteorological Society

Artificial / Absorbing Boundaries



Low-Order ABCs





Late 70's – mid 80's:

Absorbing Boundary Conditions (ABCs)

Other names: Non-reflecting, Radiating, Open, Silent, Transmitting, Transparent, Free-space, Pulled-back, One-way BCs…

Low-order (local) ABCs:

Engquist & Majda (1977), Bayliss & Turkel (1980), Kriegsmann et al. (1980), Feng (1983), Higdon (1986), …

BE, Texas

AM, Courant

AB, Northwestern

ET, TAU

GK, NJIT

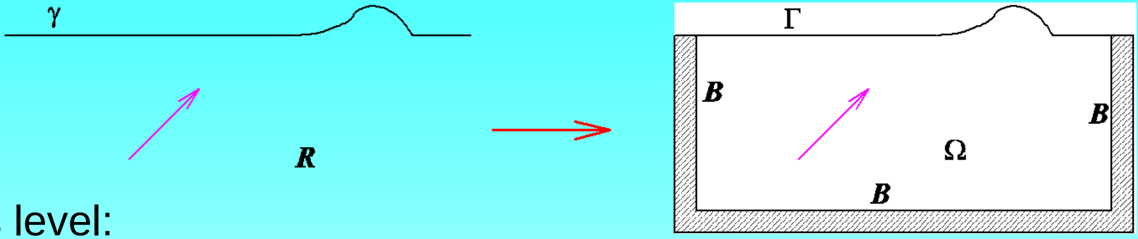
KF, Nanjing U.

RH, Oregon State U.

Two milestones

Perfectly Matched Layer (PML)

Invented by J.P. Bérenger, 1994



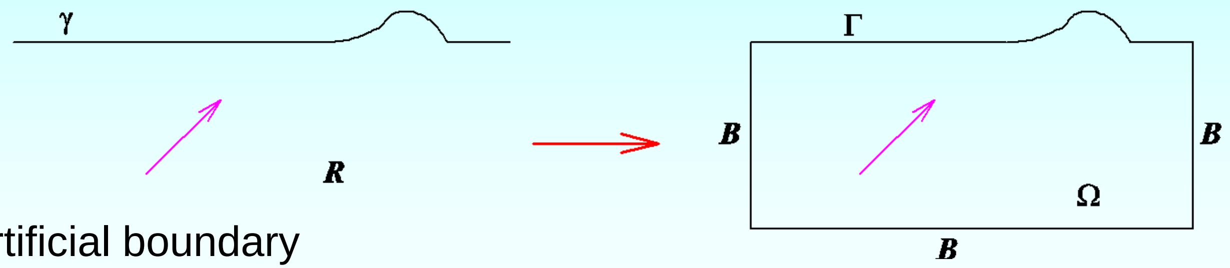
Properties at the continuous level:

- Zero reflection at the interface B for *any* plane wave
- Waves quickly damped inside the layer

Technique: modification of governing equations in the layer

High-Order ABCs

Invented by F. Collino, 1993



- Local ABC on an artificial boundary
- Accuracy (order) of ABC is arbitrarily high
- Only low-order derivatives appear

Technique: using auxiliary variables to eliminate high derivatives

ABCs and PMLs: Saul's contributions

Abarbanel, Gottlieb & Hesthaven, Non-linear PML equations for time dependent electromagnetics in three dimensions, JSC, 2006

Abarbanel, Stanescu & Hussaini, Unsplit variables PMLs for the shallow water equations with Coriolis forces, Comp. Geoph., 2003

Abarbanel, Gottlieb & Hesthaven, Long Time Behavior of the PML Equations in Computational Electromagnetics, JSC, 2002

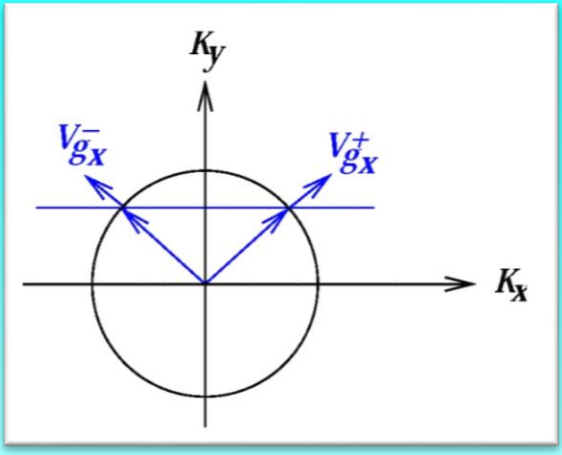
Tsynkov, Abarbanel *et al*., Global artificial boundary conditions for computation of external flows with jets, AIAA J., 2000

Abarbanel, Gottlieb & Hesthaven, Well-posed perfectly matched layers for advective Acoustics, JCP, 1999

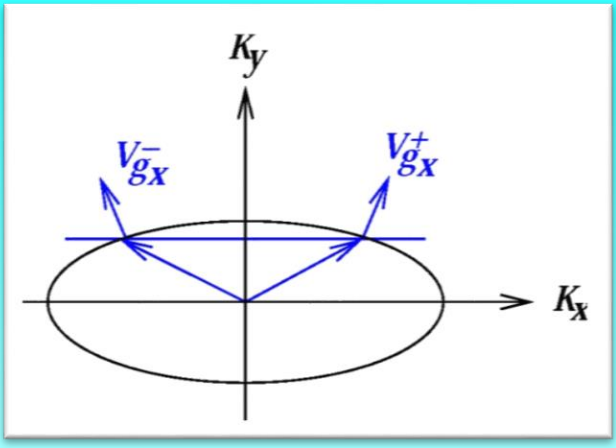
Abarbanel & Gottlieb, A mathematical analysis of the PML method, JCP, 1997

Tsynkov, Turkel & Abarbanel, External flow computations using global boundary conditions, AIAA J., 1996

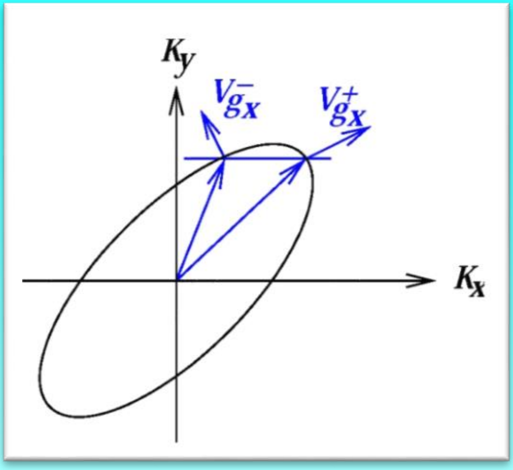
The Challenge of "Inverse Modes"



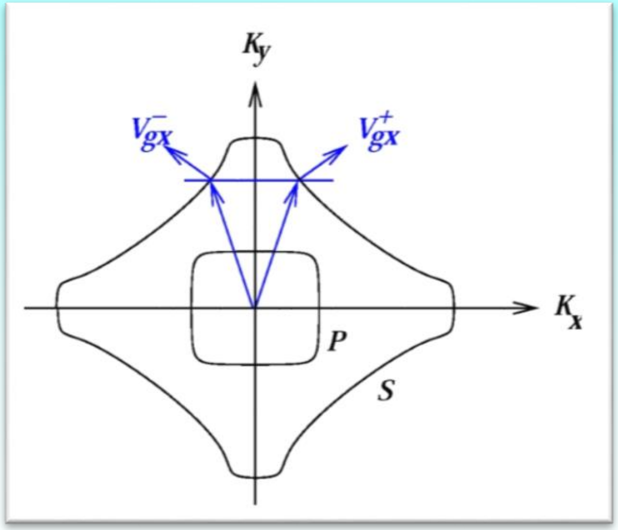
Acoustic, isotropic



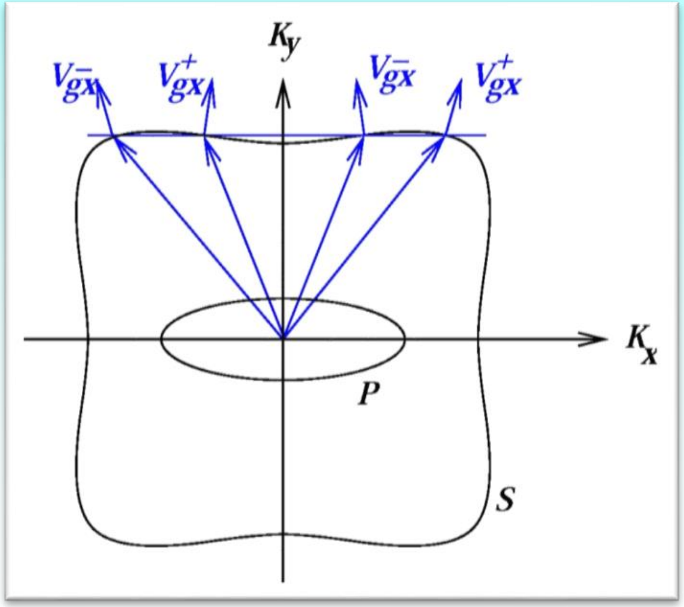
Acoustic, orthotropic,
no inverse modes



Acoustic, gen. anisotropy,
controlled inverse modes



Elastic, weakly-orthotropic,
no inverse modes



Elastic,
Strongly-orthotropic,
uncontrolled inverse
modes

The Challenge of “Inverse Modes” (Contd.)

See movie showing wave propagation with and without inverse modes

Why do standard ABCs generally fail in the presence of inverse modes?

Take, e.g., the simplest ABC in acoustics (*c*=wave speed, *x*=outward normal direction to the boundary):

$$\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}\right)u=0$$

This ABC is satisfied by waves whose phase velocity is in the outgoing direction.

Suppose an outgoing inverse-mode approaches the boundary:
energy is propagating out = group velocity is pointing out
→ phase velocity is pointing in.
The ABC would “identify” it as incoming, and would not let it out!
→ Instability

Designing ABCs (our lesson)

The standard approach:

Design the ABC based on accuracy. Then worry about stability.

Our recommended approach:

Design the ABC based on (E-) stability. Then worry about accuracy, by optimizing the ABC free parameters.

Stability Analysis (continuous level)

Standard stability analysis for hyperbolic IBV problems: the Kreiss theory [1970; book by Gustafsson, Kreiss & Oliger, 1995].
Continuous-level and discrete-level versions.
If a 1st-order system is Kreiss-stable, one gets a stability estimate of the form

$$\|u(x,t)\|_{L_2} \leq K(t)\|u(x,0)\|_{(1)}$$

Note: *K(t)* may be even exponentially growing!

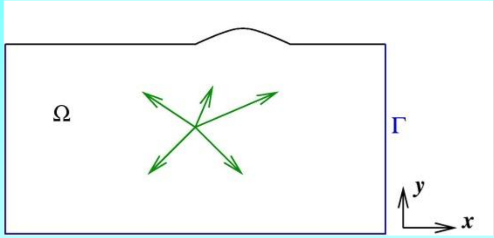
A stronger type of stability is energy-stability, based on the existence of a positive “energy function” *E[u](t)* such that *d/dt E[u](t) ≤ 0.*
From *E[u](t)≤E[u](0)* we can obtain a stability estimate which is uniform in time.

ABC for Isotropic Elasticity

Lysmer Kuhlemeyer

The Lysmer-Kuhlemeyer (LK) ABC [1969]:

$$T_x + \rho c_L \frac{\partial u_x}{\partial t} = 0 \quad , \quad T_y + \rho c_T \frac{\partial u_y}{\partial t} = 0 \qquad \text{on } \Gamma$$



The “dashpot model”.

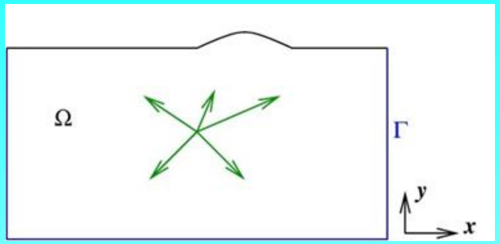
Written in terms of the medium velocities:

$$\mathbf{L}_K \mathbf{u} \equiv \left\{ \begin{bmatrix} c_L^2 & 0 \\ 0 & c_T^2 \end{bmatrix} \frac{\partial}{\partial x} + \begin{bmatrix} 0 & c_L^2 - 2c_T^2 \\ c_T^2 & 0 \end{bmatrix} \frac{\partial}{\partial y} + \begin{bmatrix} c_L & 0 \\ 0 & c_T \end{bmatrix} \frac{\partial}{\partial t} \right\} \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma$$

Exact for P and S waves at normal incidence.

1st order accuracy.

Stability of the LK ABC



Define the energy (physical)

$$E[\mathbf{u}(t)] = \frac{1}{2}\int_\Omega \left\{ \left|\frac{\partial \mathbf{u}}{\partial t}\right|^2 + (c_L^2 - 2c_T^2)\left|\nabla\cdot\mathbf{u}\right|^2 + 2c_T^2\varepsilon_{ij}\varepsilon_{ij} \right\} d\Omega$$

Differentiating w.r.t. *t*, using the elastic equations, IBP and substituting the LK condition results in

$$\frac{d}{dt}E[\mathbf{u}(t)] = -\int_\Gamma \frac{\partial \mathbf{u}}{\partial t}\cdot\begin{bmatrix} c_L & 0 \\ 0 & c_T \end{bmatrix}\frac{\partial \mathbf{u}}{\partial t}\, d\Gamma \le 0$$

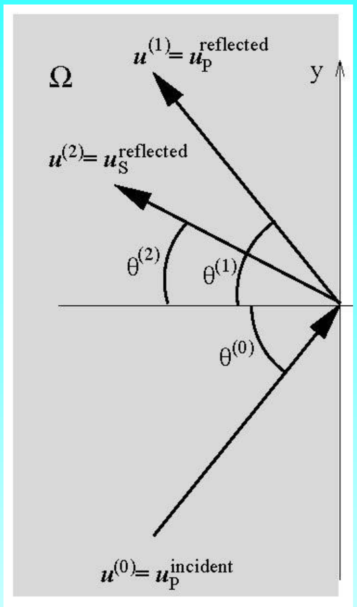
$$\Rightarrow \; E[\mathbf{u}(t)] \le E[\mathbf{u}(0)]$$

$$\Rightarrow \; \left\|D\mathbf{u}\right\|^2 \le C(\left\|\dot{\mathbf{u}}_0\right\|^2 + \left\|\nabla\mathbf{u}_0\right\|^2)$$

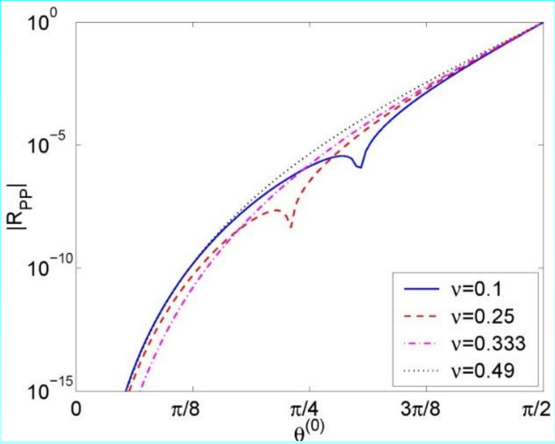
D = any first derivative*. C* does not depend on *T.*

→ Stable, uniformly in time.

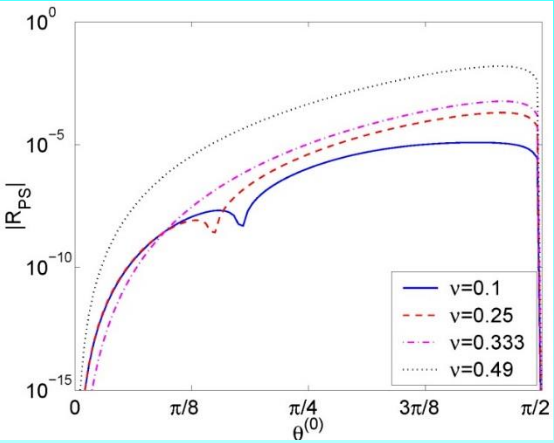
Accuracy: amplitude reflection coefficients

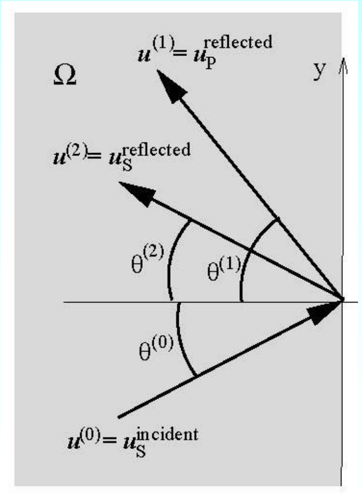


R_{PP}

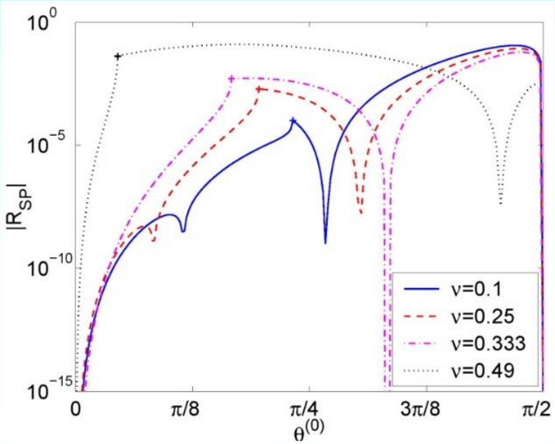


R_{PS}

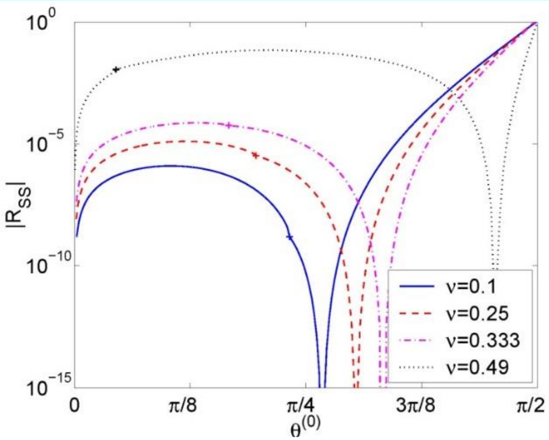




R_{SP}

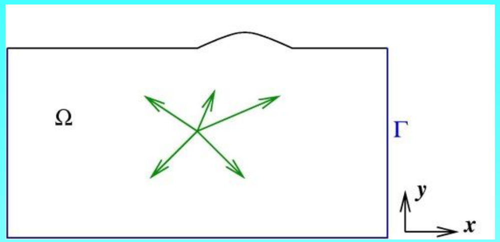


R_{SS}



Extended ABC for Anisotropic Elasticity

$$\boxed{T_i + \gamma_{ij}\dot{u}_j + \beta_{ij}u_j + \alpha_{ij}\ddot{u}_j - \zeta_{ij}u_{j,yy} - \eta_{ij}\dot{u}_{j,yy} - \xi_{ij}\ddot{u}_{j,yy} = 0}$$



where all matrices are symmetric and at least one of them is positive definite.

To obtain stability, define the non-physical "energy"

$$E[\boldsymbol{u}(t)] = E_{\text{elast}} + \frac{1}{2}\int_{\Gamma}(u_i\beta_{ij}u_j + \dot{u}_i\alpha_{ij}\dot{u}_j + u_{i,y}\zeta_{ij}u_{j,y} + \dot{u}_{i,y}\xi_{ij}\dot{u}_{j,y})d\Gamma$$

Then we can show

$$\frac{d}{dt}E[\boldsymbol{u}(t)] = -\int_{\Gamma}(\dot{u}_i\gamma_{ij}\dot{u}_j + \dot{u}_{i,y}\eta_{ij}\dot{u}_{j,y})d\Gamma \le 0 \quad \Rightarrow \quad \text{energy-stable}$$

Weak form:

Find $u \in S$ s.t. ICs are satisfied, and $\forall w \in S$

$$\int_{\Omega} w_i\rho\ddot{u}_i d\Omega + \int_{\Omega} w_{i,j}C_{ijkl}u_{k,l}d\Omega$$

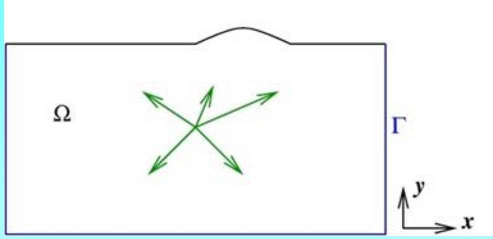
$$+\int_{\Gamma}(w_i\gamma_{ij}\dot{u}_j + w_i\beta_{ij}u_j + w_i\alpha_{ij}\ddot{u}_j - w_{i,y}\zeta_{ij}u_{j,y} - w_{i,y}\eta_{ij}\dot{u}_{j,y} - w_{i,y}\xi_{ij}\ddot{u}_{j,y})d\Gamma = \int_{\Omega} w_i f_i d\Omega$$

$\Rightarrow$ Symmetric FE formulation

The Energy-Rate Reflection Coefficient (ERRC)

In the anisotropic case, due to the presence of inverse modes, amplitude RC's (related to phase velocity) are not meaningful.

Need to base the RC on energy or energy-rate (related to group velocity).



Plane waves in an anisotropic medium:

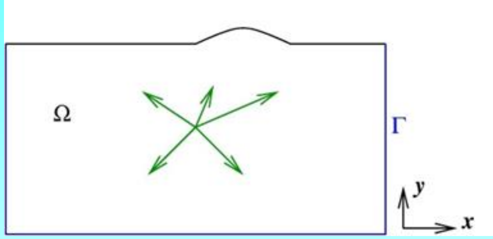
$$\boldsymbol{u}(\boldsymbol{x}, t) = A\boldsymbol{D}e^{i(-kx\cos\theta - ky\sin\theta + \omega t)},$$

Where ω is related to k through the dispersion relation,
$\boldsymbol{D}$ is the eigenvector corresponding to ω, A is the amplitude,
and θ is the angle of incidence.
No pure P and S waves, but quasi-P and quasi-S waves.

The Energy-Rate Reflection Coefficient (ERRC), Contd.

The elastic energy:

$$E = \frac{1}{2}\int_{\Omega} \left(\rho u_{i,t}\bar{u}_{i,t} + C_{ijkl}u_{i,j}\bar{u}_{k,l}\right) d\,\Omega,$$



From this we get the energy rate:

$$\frac{dE}{dt} = \frac{1}{2}\int_{\Gamma} C_{ijkl}\left(\bar{u}_{i,t}u_{k,l}n_j + u_{i,t}\bar{u}_{k,l}n_j\right) d\,\Gamma,$$

Substituting the plane wave expression yields, after some algebra,

$$\frac{dE}{dt} = \int_{\Gamma} C_{ixkl}|A|^2\omega k D_i D_k P_l \, d\Gamma,$$

where ***P*** is an "indicator" unit vector that determines whether the wave is an inverse mode or not.

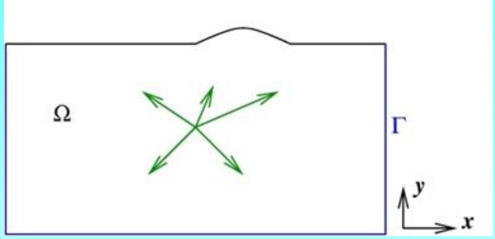
Take the integrand as the basis for the ERRC

The Energy-Rate Reflection Coefficient (ERRC), Contd.

The energy-rate density:

$$\mathrm{ERD} = C_{ixkl} \, |A|^2 \, \omega k D_i D_k P_l$$

The 4 ERRC's are defined by

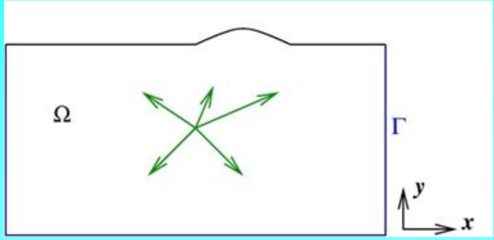


$$\mathrm{ERRC}_{mn} = \frac{\mathrm{ERD}_n^{\mathrm{reflected}}}{\mathrm{ERD}_m^{\mathrm{incident}}} \quad , \; m,n = \mathrm{P \text{ or } S}$$

These ERRCs depend on:

(1) the given incident angle θ^0,
(2) the given material properties C_{ixkl} ,
(3) the amplitude-RCs A, which are computed in the usual way (as if there are no inverse modes); depend on the free parameters in the ABC.

Optimization



For a given angle of incidence calculate

$$\mathrm{ERRC}_{mn}(\theta^0) = \frac{\mathrm{ERD}_n^{\mathrm{reflected}}}{\mathrm{ERD}_m^{\mathrm{incident}}} \quad , \; m,n = \mathrm{P\ or\ S}$$

Then define the cost function

$$W = \max_{m,n} \int_0^{\pi/2} w(q,\theta^0)\,|\,\mathrm{ERRC}_{m.n}(\theta^0)\,|\,d\theta^0$$

$$\text{where} \quad w(q,\theta^0) = \exp[\mathrm{q}(1-1/\cos\theta^0)] \quad , \quad q \geq 0$$

The weighting function *w* attributes more importance to close-to-normal waves than to oblique waves (unless *q=0*)

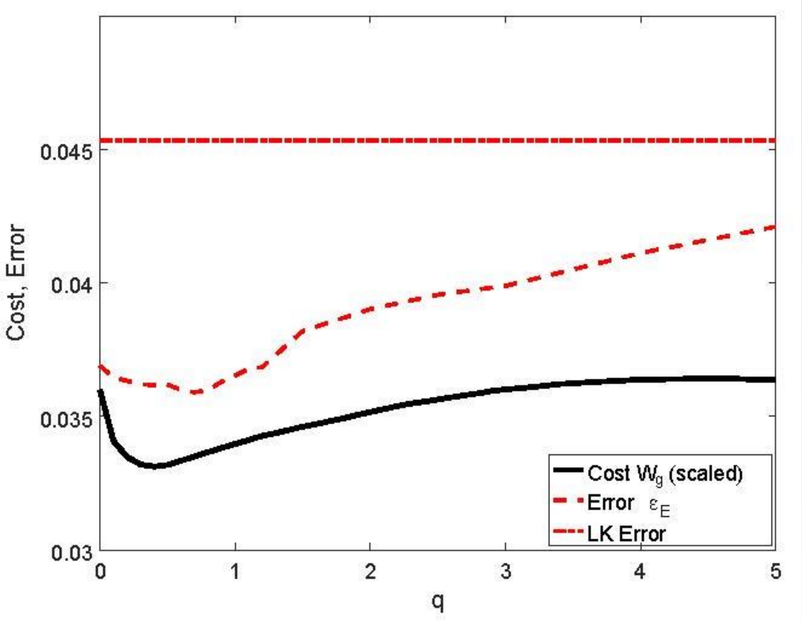
The optimization is done using a genetic algorithm, to avoid a local-minimum trap.

Numerical Example

Orthotropic material

Only the traction and γ_{ij} terms are taken in the ABC

Optimal ABC gives a maximal error which is smaller by ~20% than the LK error

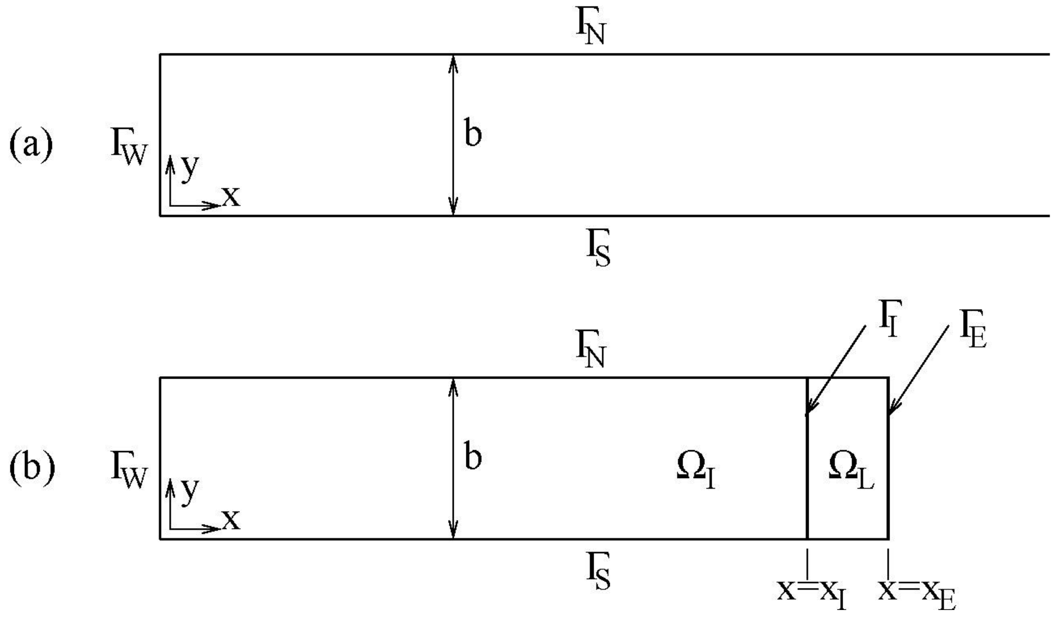


See movie showing solutions and errors

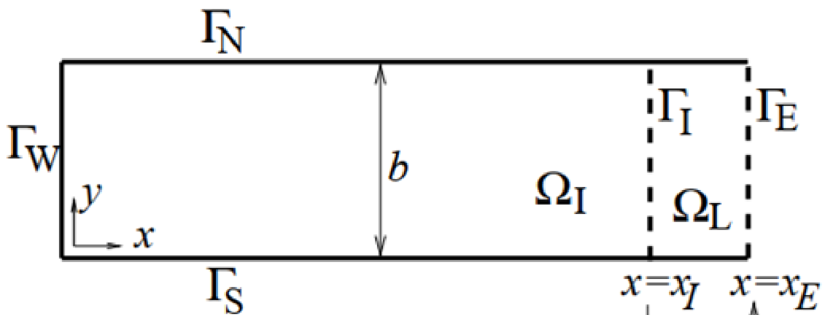
The Double Absorbing Boundary Method (DAB)

- A new approach for solving wave problems in unbounded domains.
- Common features to local high-order Absorbing Boundary Conditions (ABC) and Perfectly Matched Layers (PML).
- Enjoys relative advantages with respect to both.

- Idea: Require each auxiliary variable to satisfy the wave equation in the layer; apply the high-order ABC on both inner and outer boundaries of the layer.



The DAB setup for elastodynamics

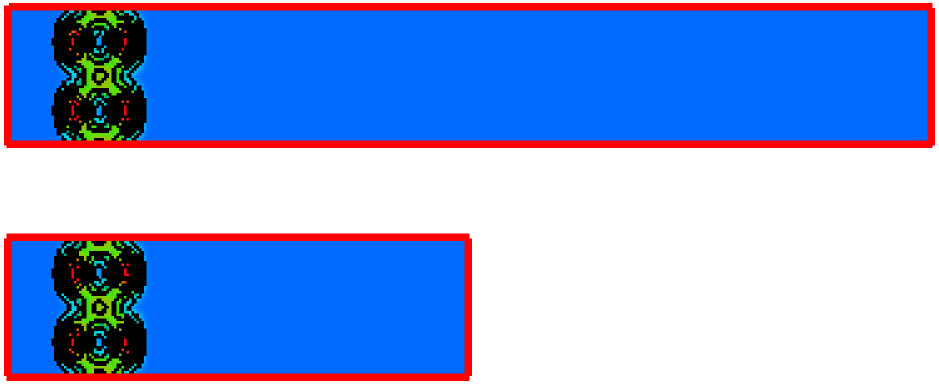


$$
\begin{aligned}
a_0\dot{\phi}^{(1)} - c_L\phi^{(1)}_{,x} &= a_0\dot{\phi}^{(0)} + c_L\phi^{(0)}_{,x} \\
a_1\dot{\phi}^{(2)} - c_L\phi^{(2)}_{,x} &= a_1\dot{\phi}^{(1)} + c_L\phi^{(1)}_{,x} \\
&\vdots \\
a_{P-1}\dot{\phi}^{(P)} - c_L\phi^{(P)}_{,x} &= a_{P-1}\dot{\phi}^{(P-1)} + c_L\phi^{(P-1)}_{,x}
\end{aligned}
$$

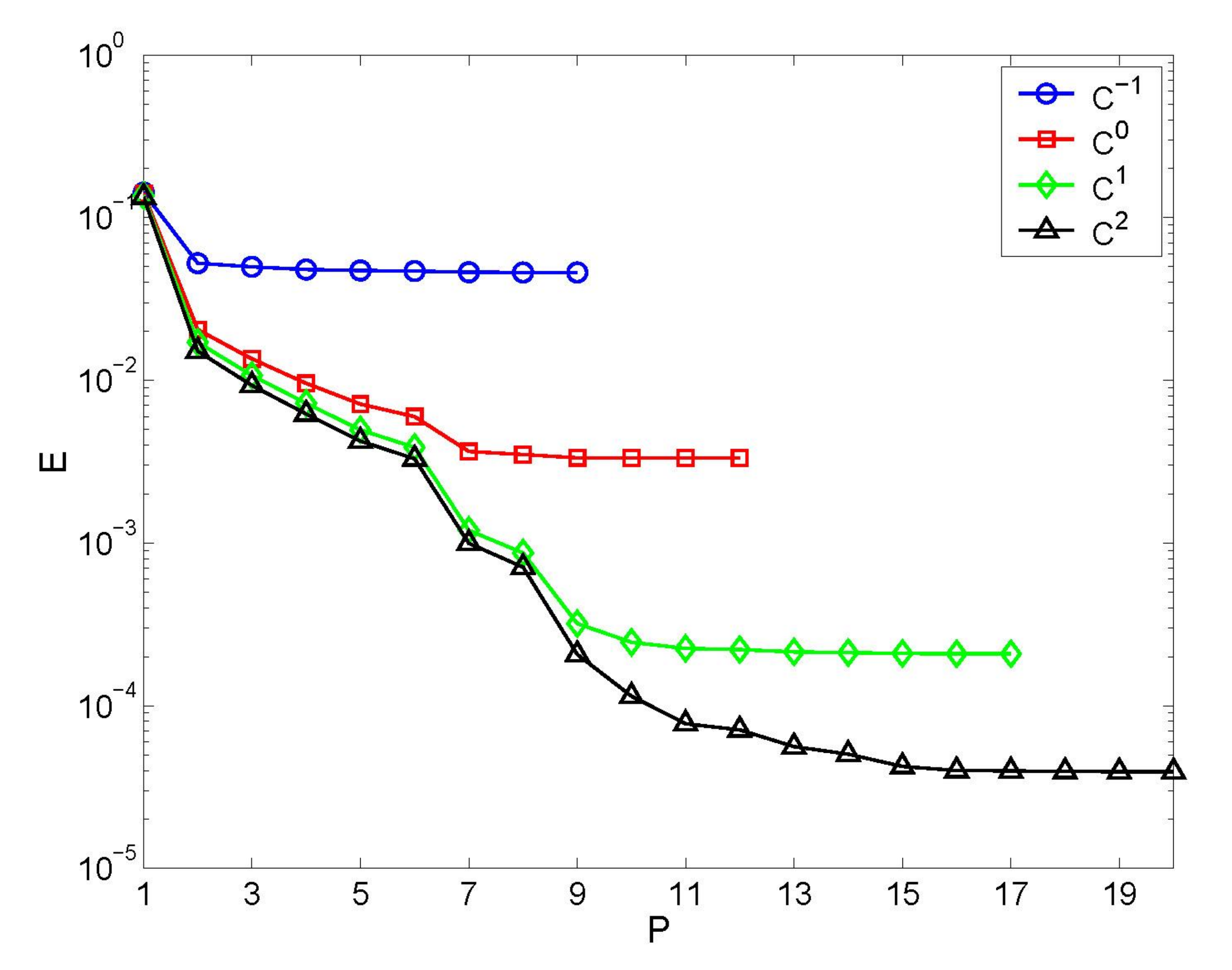
$$
\begin{aligned}
a_0\dot{\phi}^{(0)} + c_L\phi^{(0)}_{,x} &= a_0\dot{\phi}^{(1)} - c_L\phi^{(1)}_{,x} \\
a_1\dot{\phi}^{(1)} + c_L\phi^{(1)}_{,x} &= a_1\dot{\phi}^{(2)} - c_L\phi^{(2)}_{,x} \\
&\vdots \\
a_{P-1}\dot{\phi}^{(P-1)} + c_L\phi^{(P-1)}_{,x} &= a_{P-1}\dot{\phi}^{(P)} - c_L\phi^{(P)}_{,x} \\
T^{(P)}_x + \rho c_L\dot{\phi}^{(P)}_x &= 0 \\
T^{(P)}_y + \rho c_T\dot{\phi}^{(P)}_y &= 0
\end{aligned}
$$

Elastic waveguide – 2D example movie

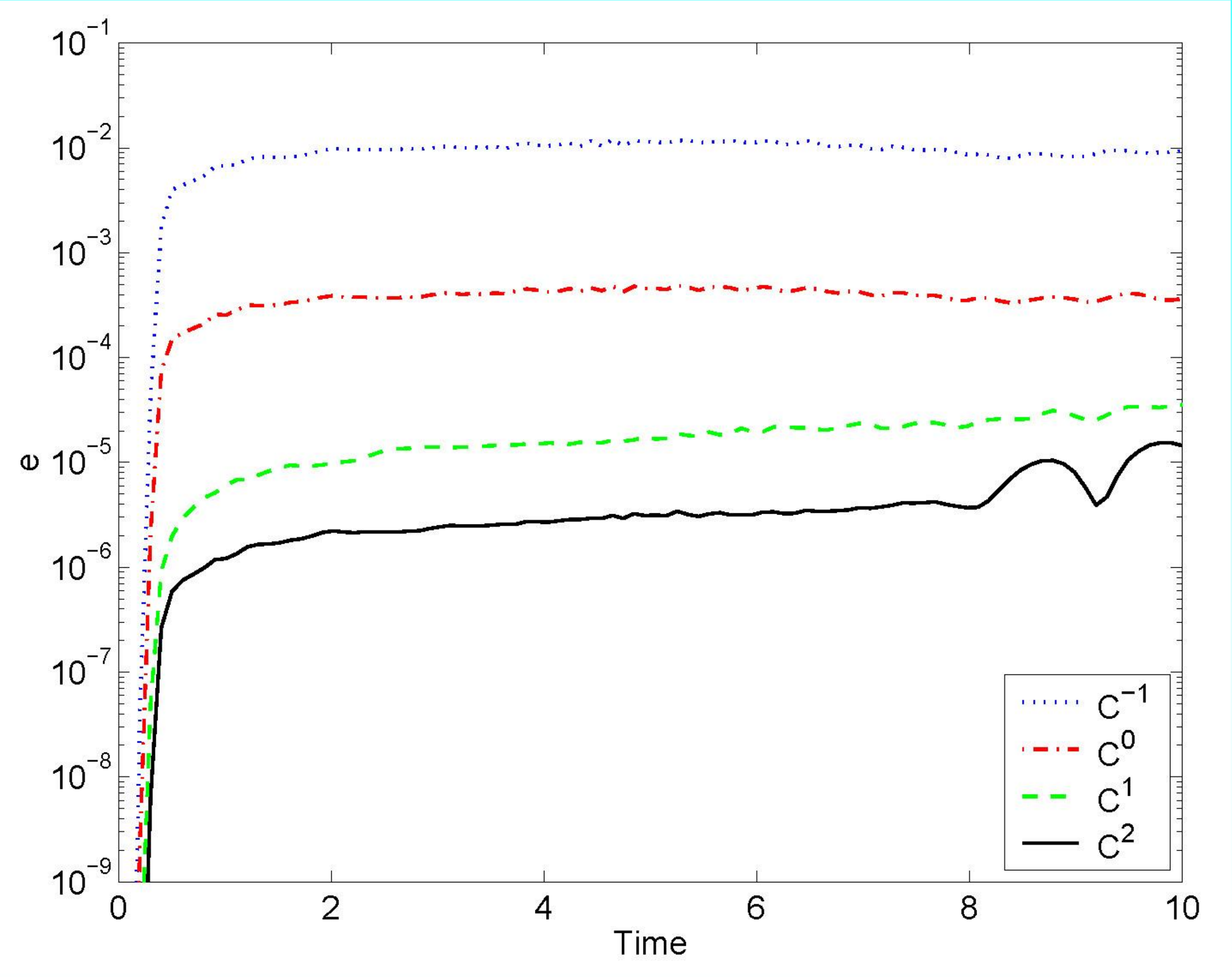
P=10



Elastic waveguide 2D example – contd.



Elastic waveguide 2D example – contd.



Layered Medium

- Extension of DAB to a layered medium, where the layer interfaces are normal to the boundary, is very easy. It can be shown that the same formulation applies as in the case of a homogeneous medium.

- Jump conditions across layer interfaces for all auxiliary variables:

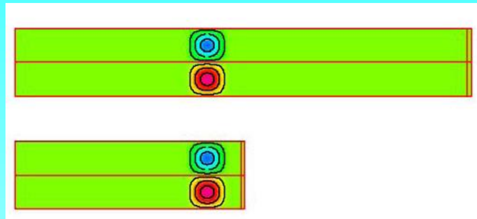
$$[[\phi_{jx}]] = 0 \quad , \quad [[\phi_{jy}]] = 0 \qquad \text{(displacement continuity)}$$

$$[[\mathrm{T}_{jx}]] = 0 \quad , \quad [[\mathrm{T}_{jy}]] = 0 \qquad \text{(traction continuity)}$$

These conditions are enforced weakly (as natural interface conditions) in a FE formulation.

Layered Medium, example

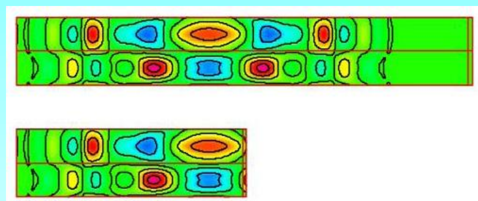
t=0



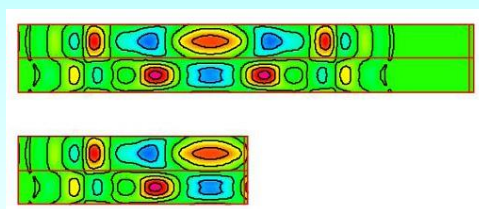
← Reference solution

← Solution with DAB

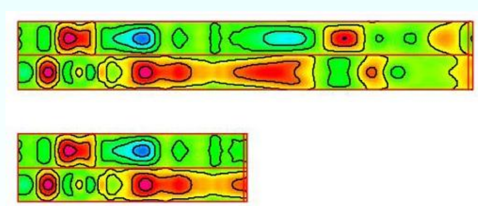
t=2



t=4



t=10



Anisotropic Medium, example

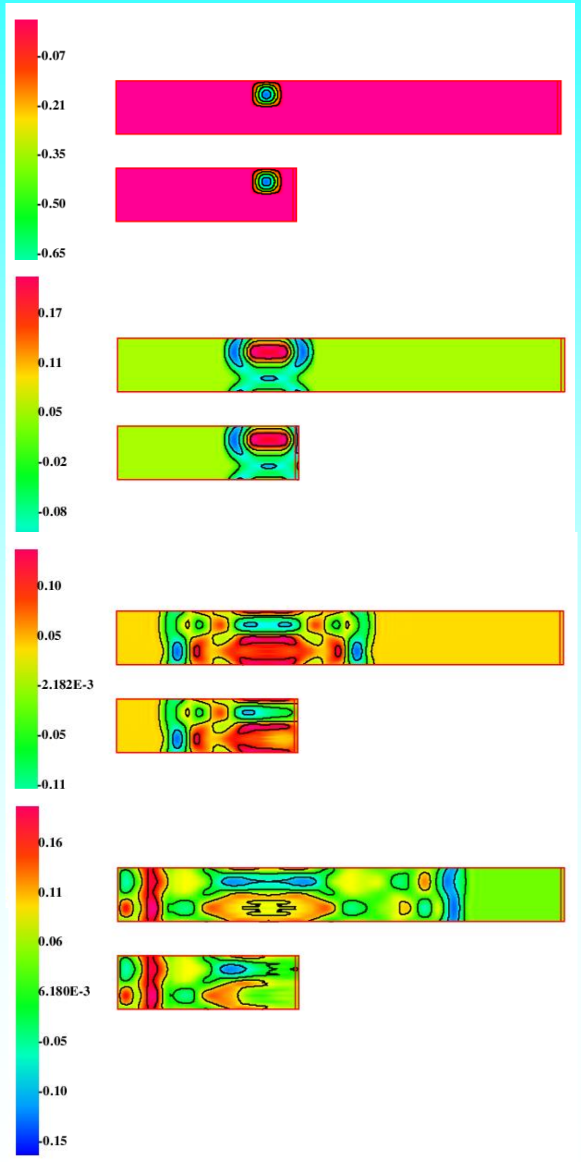
$$c_{11}=3,\quad c_{22}=2,\quad c_{12}=0.25,\quad c_{33}=1,\quad c_{13}=c_{23}=0$$

t=0

t=2

t=6

t=12



← Reference solution

← Solution with DAB

Extensions

- Energy-stable high-order DAB formulations for acoustics and elastodynamics
- Optimal high-order ABCs
- Adaptive optimal ABCs
- Implement for more realistic geophysical configurations