In memoriam of Professor Saul Abarbanel

An Embedded Cartesian Scheme for the Navier-Stokes Equations

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Outline

1. Navier-Stokes equations in streamfunction formulation
2. The one dimensional problem
3. Fourth order schemes in 2D regular domains
4. Fourth-order schemes for the N-S problem in irregular domains
5. Eigenvalues and Eigenfunctions of Biharmonic Problems

Navier-Stokes Equations in Pure Streamfunction Formulation (Lagrange 1768)

Let $\mathbf{u}(\mathbf{x},t) = \nabla^{\perp}\psi$, where ψ is the streamfunction. Then

$$\partial_t(\Delta\psi) + (\nabla^{\perp}\psi)\cdot\nabla(\Delta\psi) = \nu\Delta^2\psi, \quad \text{in } \Omega.$$

The boundary and initial conditions are

$$\psi(x,y,t) = \frac{\partial\psi}{\partial n}(x,y,t) = 0, \quad (x,y)\in\partial\Omega,$$

$$\psi_0(x,y) = \psi(x,y,t)|_{t=0}, \quad (x,y)\in\Omega.$$

There is no need for vorticity boundary conditions.

(*) Goodrich-Gustafson-Halasi, JCP (1990).

[1] M. Ben-Artzi, J.-P. Croisille, D. Fishelov and S. Trachtenberg, J. Comp. Phys. 2005.

Approximation in the one-dimensional case

Consider the problem

$$\begin{cases} \psi^{(4)}(x) = f(x), \quad 0 < x < 1 \\ \psi(0) = 0,\ \psi(1) = 0,\ \psi'(0) = 0,\ \psi'(1) = 0. \end{cases} \tag{1}$$

We lay out a uniform grid $x_0, x_1, ..., x_N$ where $x_i = ih$ and $h = 1/N$.
We approximate ψ on $[x_{i-1}, x_{i+1}]$ by a polynomial of degree 4,

$$Q(x) = a_0 + a_1(x - x_i) + a_2(x - x_i)^2 + a_3(x - x_i)^3 + a_4(x - x_i)^4,$$

with interpolating values

$$\psi_{i-1}, \psi_i, \psi_{i+1}, \psi_{x,i-1}, \psi_{x,i+1},$$

where $\psi_{x,i-1}, \psi_{x,i+1}$ are approximate values for $\psi'(x_{i-1}), \psi'(x_{i+1})$, which will be determined by the system as well.

Approximation in the one-dimensional case

We obtain

$$\begin{cases} (a) & a_0 = \psi_i, \\ (b) & a_1 = \frac{3}{2}\delta_x\psi_i - \frac{1}{4}(\psi_{x,i+1} + \psi_{x,i-1}), \\ (c) & a_2 = \delta_x^2\psi_i - \frac{1}{2}(\delta_x\psi_x)_i, \\ (d) & a_3 = \frac{1}{h^2}\left[\frac{1}{4}(\psi_{x,i+1} + \psi_{x,i-1}) - \frac{1}{2}\delta_x\psi_i\right] \\ (e) & a_4 = \frac{1}{2h^2}\left((\delta_x\psi_x)_i - \delta_x^2\psi_i\right). \end{cases} \tag{2}$$

Approximation in the one-dimensional case

The approximate value $\psi_{x,i}$ is chosen as $Q'(x_i)$. Thus,

$$\psi_{x,i} \overset{def}{=} a_1 = \frac{3}{2}\delta_x\psi_i - \frac{1}{4}(\psi_{x,i+1} + \psi_{x,i-1}).$$

This yields the Padé approximation

$$\frac{1}{6}\psi_{x,i-1} + \frac{2}{3}\psi_{x,i} + \frac{1}{6}\psi_{x,i+1} = \delta_x\psi_i, \quad 1 \le i \le N-1. \tag{3}$$

A natural approximation to $\psi^{(4)}(x_i)$ is therefore $Q^{(4)}(x_i)$. Thus,

$$\delta_x^4\psi_i \overset{def}{=} 24a_4 = \frac{12}{h^2}\left((\delta_x\psi_x)_i - \delta_x^2\psi_i\right). \tag{4}$$

Approximation in the one-dimensional case

An approximation for the one-dimensional biharmonic problem is

$$\begin{cases} (a) \qquad \delta_x^4 \tilde{\psi}_i = f(x_i) \quad 1 \le i \le N-1, \\ (b) \qquad \sigma_x \tilde{\psi}_{x,i} = \delta_x \tilde{\psi}_i, \quad 1 \le i \le N-1, \\ (c) \qquad \tilde{\psi}_0 = 0,\ \tilde{\psi}_N = 0,\ \tilde{\psi}_{x,0} = 0,\ \tilde{\psi}_{x,N} = 0. \end{cases} \tag{5}$$

where

$$\sigma_x \varphi = \frac{1}{6}\varphi_{i-1} + \frac{2}{3}\varphi_i + \frac{1}{6}\varphi_{i+1}.$$

Consistency of the three-point biharmonic operator

Proposition

Suppose that $\psi(x)$ is a smooth function on $[0,1]$. Then,

- $$|\sigma_x\big(\delta_x^4\psi_i^* - (\psi^{(4)})^*(x_i)\big)| \leq Ch^4\|\psi^{(8)}\|_{L^\infty}, \ \ 2 \leq i \leq N-2. \qquad (6)$$

- *At near boundary points $i=1$ and $i=N-1$, the fourth order accuracy of (6) drops to first order,*

$$|\sigma_x\big(\delta_x^4\psi_i^* - (\psi^{(4)})^*(x_i)\big)| \leq Ch\|\psi^{(5)}\|_{L^\infty}, \ \ i=1,N-1. \qquad (7)$$

Optimal convergence of the three-point biharmonic operator

The following error estimate holds.

Theorem

Let $\tilde{\psi}$ be the approximate solution of the biharmonic problem and let ψ be the exact solution and ψ^* its evaluation at grid points. The error $\mathfrak{e} = \tilde{\psi} - \psi^* = \delta_x^{-4} f^* - (\partial_x^{-4} f)^*$ satisfies

$$\max_{1 \leq i \leq N-1} |\mathfrak{e}_i| \leq C h^4, \quad |\mathfrak{e}|_h \leq C h^4, \tag{8}$$

where C depends only on f.

[2] M. Ben-Artzi, J.-P. Croisille and D. Fishelov, Navier-Stokes Eqns. in Planar Domains, 2013, Imperial College Press. J. Scientific Computing, 2012.
B. Gustafsson,1981,S. Abarbanel, A. Ditkowski and B. Gustafsson,2000, M. Svard and J. Nordstrom,2006

Linear time-independent equation- constant coefficients case

Consider an invertible problem

$$u^{(4)} + au^{(2)} + bu = f, \quad x \in [0,1], \tag{9}$$

(with boundary conditions on u, u') and its approximation

$$\delta_x^4 \mathfrak{v} + a\tilde{\delta}_x^2 \mathfrak{v} + b\mathfrak{v} = f^*, \tag{10}$$

where $\tilde{\delta}_x^2 \mathfrak{v} = 2a_2 = 2\delta_x^2 \mathfrak{v} - \delta_x \mathfrak{v}_x$. Then, the error $\mathfrak{e} = \mathfrak{v} - u^*$ satisfies

$$|\mathfrak{e}(t)|_h \leq Ch^4, \tag{11}$$

where $C > 0$ depends only on f.
[3] M. Ben-Artzi, J.-P. Croisille, D. Fishelov and R. Katzir, IMA J. Numer. Anal, 2017.

The linear evolution equation

Consider

$$\partial_t u = -\partial_x^4 u + a\partial_x^2 u + bu, \quad x \in [0,1], \qquad t \geq 0. \tag{12}$$

with the initial condition $u(t=0) = u_0$, and its approximation

$$\mathfrak{v}_t = -\delta_x^4 \mathfrak{v} + a\tilde{\delta}_x^2 \mathfrak{v} + b\mathfrak{v}, \qquad t \geq 0. \tag{13}$$

Then the error $\mathfrak{e} = \mathfrak{v} - u^*$ satisfies

$$|\mathfrak{e}(t)|_h \leq Ch^{4-\epsilon}, \quad t \in [0,T], \; h < h_0, \tag{14}$$

where $C > 0$ depends only on u_0, T, ϵ.
[4] M. Ben-Artzi, J.-P. Croisille and D. Fishelov, submitted.

The linear evolution equation-sketch of the proof for $u_t = -u_{xxxx}$

Consider

$$\partial_t u = -\partial_x^4 u, \quad x \in [0,1], \qquad t \geq 0. \tag{15}$$

Applying ∂_x^{-4} on the last equation,

$$\partial_t \partial_x^{-4} u = -u. \tag{16}$$

By the optimal error bound for $\partial_x^{-4}\partial_t u = -u$ we have

$$\partial_t \delta_x^{-4} u^* = -u^* + O(h^4). \tag{17}$$

Consider the approximation $\partial_t \mathfrak{v} = -\delta_x^4 \mathfrak{v}$ and applying δ_x^{-4} on the last equation, we have

$$\partial_t \delta_x^{-4} \mathfrak{v} = -\mathfrak{v}. \tag{18}$$

Then the error $\mathfrak{e} = \mathfrak{v} - u^*$ satisfies

$$\partial_t \delta_x^{-4} \mathfrak{e}(t) = -\mathfrak{e}(t) + O(h^4). \tag{19}$$

The linear evolution equation-sketch of the proof

Defining $\mathfrak{w} = \delta_x^{-4}\mathfrak{e}$

$$\partial_t \mathfrak{w}(t) = -\mathfrak{e}(t) + O(h^4). \tag{20}$$

Inner multiplication with $\mathfrak{w}(t)$ yields

$$\frac{1}{2}\partial_t |\mathfrak{w}(t)|_h^2 + (\mathfrak{e}(t), \mathfrak{w}(t))_h = (O(h^4), \mathfrak{w}(t))_h. \tag{21}$$

By the coercivity $(\mathfrak{e}(t), \mathfrak{w}(t))_h = (\delta_x^4 \mathfrak{w}, \mathfrak{w})_h \geq C|\mathfrak{w}(t)|_h^2$

$$\partial_t |\mathfrak{w}(t)|_h^2 + C|\mathfrak{w}(t)|_h^2 \leq O(h^8) + |\mathfrak{w}(t)|_h^2. \tag{22}$$

By Grownwall's inequality $|\mathfrak{w}(t)|_h \leq Ch^4$.

The linear evolution equation-sketch of the proof

Going back to

$$\partial_t \mathfrak{w}(t) = -\mathfrak{e}(t) + O(h^4). \tag{23}$$

Approximating $\partial_t \mathfrak{w}(t)$ by a finite difference scheme $S_Q \mathfrak{w}$, for which $S_Q \mathfrak{w}(t) - \mathfrak{w}'(t) = O((\Delta t)^Q)$, and choosing $\Delta t = h^{4/Q} = h^\epsilon$,

$$|\mathfrak{e}(t)|_h \leq C h^{4-\epsilon}, \quad t \in [0, T], \ h < h_0, \tag{24}$$

where $C > 0$ depends only on u_0, T, ϵ.

Numerical results for time-dependent problems in 1D-Kuramuto-Sivashinsky Eqn.

Consider the Kuramoto–Sivashinsky equation

$$\begin{aligned} &\partial_t u = -\partial_x^4 u - \partial_x^2 u - u\partial_x u + f, \quad -30 < x < 30, \quad t > 0, \\ &u(0,t) = \partial_x u(0,t) = 0 = u(1,t) = \partial_x u(1,t) = 0. \end{aligned} \tag{25}$$

We pick up the exact solution $u(x,t)$ as in Xu and Shu (2006)

$$u(x,t) = c+(15/19)\sqrt{11/19}(-9\tanh(k(x-ct-x_0))+11\tanh^3(k(x-ct-x_0)). \tag{26}$$

Here $c = -0.1, k = 0.5\sqrt{11/19}$ and $x_0 = -10$.

Mesh	$N = 241$	Rate	$N = 481$	Rate	$N = 961$
$\lvert\mathfrak{e}\rvert_h$	3.2873(-4)	3.99	2.0752(-5)	4.00	1.2984(-6)
$\lvert\mathfrak{e}_x\rvert_h$	2.9822(-4)	3.95	1.9332(-5)	3.98	1.2246(-6)

Table: KS equation (25), where $t = 1$ and $\Delta t = h^2$.

Numerical results for time-dependent problems in 1D-Kuramoto-Sivashinsky Eqn.

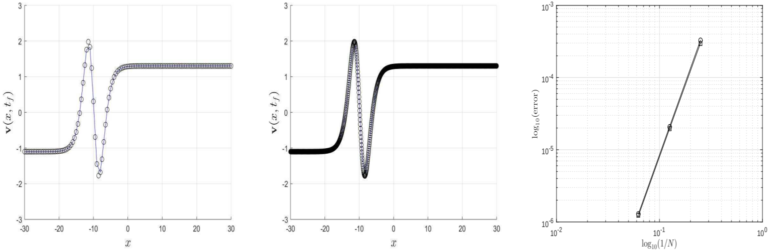


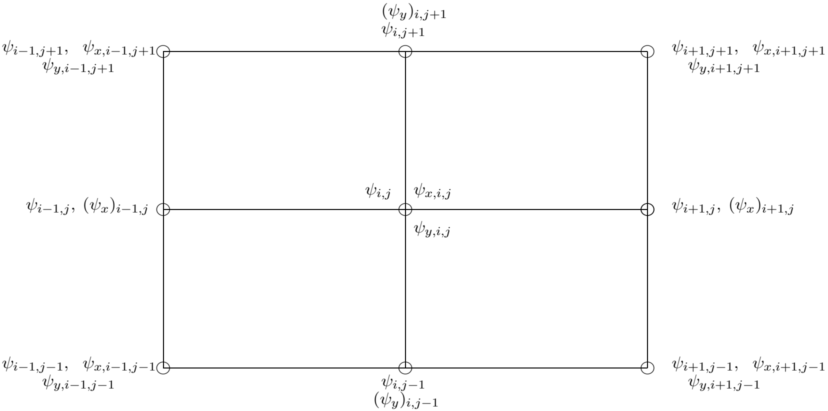
Figure: Third KS numerical example: Exact solution (solid line) and computed solution (circles) for $N = 121$ (left) and $N = 961$ (center) The convergence rate for the KS equation is documented in the right panel for u (circles) and $\frac{\partial u}{\partial x}$ (squares).

The 2D NS Equation
Discretization of the biharmonic operator

Suppose the differential problem is given for x, y in $\Omega = [0,1] \times [0,1]$ and lay out a uniform grid (x_i, y_j), $0 \le i, j \le N$.
Denoting by $\tilde{x} = x - x_i$, $\tilde{y} = y - y_j$, we approximate $\psi(x,y)$ on $[x_{i-1}, x_{i+1}] \times [y_{j-1}, y_{j+1}]$ by

$$\begin{aligned} P(x,y) &= a_0 + a_1\tilde{x} + a_2\tilde{y} + a_3\tilde{x}^2 + a_4\tilde{x}\tilde{y} + a_5\tilde{y}^2 \\ &+ a_6\tilde{x}^3 + a_7\tilde{x}^2\tilde{y} + a_8\tilde{x}\tilde{y}^2 + a_9\tilde{y}^3 \\ &+ a_{10}\tilde{x}^4 + a_{11}\tilde{x}^3\tilde{y} + a_{12}\tilde{x}^2\tilde{y}^2 + a_{13}\tilde{x}\tilde{y}^3 + a_{14}\tilde{y}^4 \\ &+ a_{15}\tilde{x}^5 + a_{16}\tilde{x}^4\tilde{y} + a_{17}\tilde{x}^3\tilde{y}^2 + a_{18}\tilde{x}^2\tilde{y}^3 + a_{19}\tilde{x}\tilde{y}^4 + a_{20}\tilde{y}^5 \\ &+ a_{21}\tilde{x}^4\tilde{y}^2 + a_{22}\tilde{x}^2\tilde{y}^4. \end{aligned} \tag{27}$$

Modified Stephenson's Scheme



J.W. Stephenson, "Single cell discretizations of order two and four for biharmonic problems", J. Comp. Phys. 1984.

Fourth Order Spatial Discretization for the biharmonic operator

$$\delta_x^4\psi_{i,j} = \frac{12}{h^2}\left\{(\delta_x\psi_x)_{i,j} - \delta_x^2\psi_{i,j}\right\} \quad , \quad 1 \le i,j \le N-1.$$

$$\delta_y^4\psi_{i,j} = \frac{12}{h^2}\left\{(\delta_y\psi_y)_{i,j} - \delta_y^2\psi_{i,j}\right\} \quad , \quad 1 \le i,j \le N-1.$$

The mixed term ψ_{xxyy} is approximated by

$$\tilde{\delta}_{xy}^2\psi_{i,j} = 3\delta_x^2\delta_y^2\psi_{i,j} - \delta_x^2\delta_y\psi_{y,i,j} - \delta_y^2\delta_x\psi_{x,i,j} = \partial_x^2\partial_y^2\psi_{i,j} + O(h^4)$$

The Laplacian of ψ is approximated by $\tilde{\Delta}_h\psi$:, where

$$\tilde{\Delta}_h\psi = 2\delta_x^2\psi - \delta_x\psi_x + 2\delta_y^2\psi - \delta_y\psi_y.$$

[5] M. Ben-Artzi, J.-P. Croisille and D. Fishelov, SISC 2008.

The Convective Term

The convective term $-\psi_y \Delta_x \psi + \psi_x \Delta_y \psi$ is therefore approximated by

$$\tilde{C}_h(\psi) = -\tilde{\psi}_y \left[\Delta_h \tilde{\psi}_x + \frac{5}{2} \left(6 \frac{\delta_x \psi - \tilde{\psi}_x}{h^2} - \delta_x^2 \tilde{\psi}_x \right) + \delta_x \delta_y^2 \psi - \delta_x \delta_y \tilde{\psi}_y \right] + \tilde{\psi}_x \left[\Delta_h \tilde{\psi}_y + \frac{5}{2} \left(6 \frac{\delta_y \psi - \tilde{\psi}_y}{h^2} - \delta_y^2 \tilde{\psi}_y \right) + \delta_y \delta_x^2 \psi - \delta_y \delta_x \tilde{\psi}_x \right],$$

where $\tilde{\psi}_x$ and $\tilde{\psi}_y$ are the 6-th order accurate Padé approximations to $\partial_x \psi$ and $\partial_y \psi$.

[6] M. Ben-Artzi, J.-P. Croisille, D. Fishelov, Navier-Stokes Equations in Planar Domains, Imperial College Press, 2013. See also J. Scientific Computing, 2009.

T. Hou and B. Wetton, 2009.

Fourth-order spatial discretization and an IMEX scheme

$$\frac{(\tilde{\Delta}_h \psi_{i,j})^{n+1/2} - (\tilde{\Delta}_h \psi_{i,j})^n}{\Delta t/2} =$$
$$-\tilde{C}_h(\psi^n)_{i,j} + \frac{\nu}{2}[\tilde{\Delta}_h^2 \psi_{i,j}^{n+1/2} + \tilde{\Delta}_h^2 \psi_{i,j}^n]$$

$$\frac{(\tilde{\Delta}_h \psi_{i,j})^{n+1} - (\tilde{\Delta}_h \psi_{i,j})^n}{\Delta t} =$$
$$-\tilde{C}_h(\psi^{n+1/2})_{i,j} + \frac{\nu}{2}[\tilde{\Delta}_h^2 \psi_{i,j}^{n+1} + \tilde{\Delta}_h^2 \psi_{i,j}^n].$$

Note here that only the discrete Laplacian and biharmonic operators, which are approximated by a compact scheme, have to be inverted at each step.

Convergence of the semi-discrete scheme

Theorem: Let $\tilde{\psi}$ be the solution of

$$\partial_t \Delta_h \widetilde{\psi} = -\nabla_h^{\perp} \widetilde{\psi} \cdot (\Delta_h \nabla_h \widetilde{\psi}) + \nu \Delta_h^2 \widetilde{\psi},$$

and ψ is the exact solution of NS equations:

$$\partial_t \Delta \psi = -\nabla^{\perp} \psi \cdot \nabla(\Delta \psi) + \nu \Delta^2 \psi.$$

Define the error $e(t)$ as $e(t) = \widetilde{\psi} - \psi$. Let $T > 0$. Then there exist constants $C, h_0 > 0$, depending possibly on T, ν and the exact solution ψ, such that, for all $0 \leq t \leq T$,

$$|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2 \leq Ch^3 \quad , \quad 0 < h \leq h_0.$$

[7] M. Ben-Artzi, J.-P. Croisille, D. Fishelov, "Convergence of a compact scheme for the pure streamfunction formulation of Navier-Stokes equations", SINUM, 2006

A High Order scheme for Irregular domains

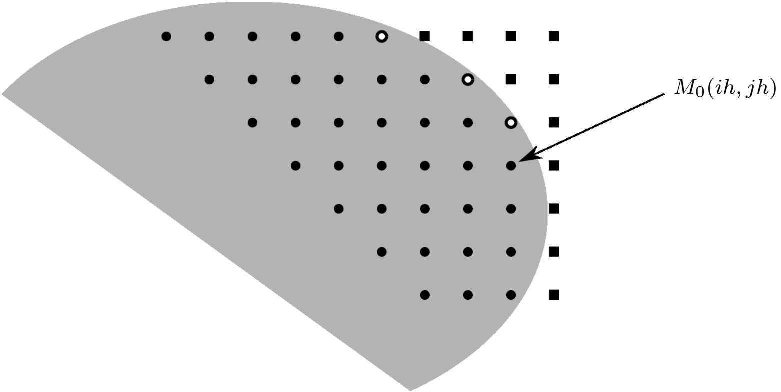
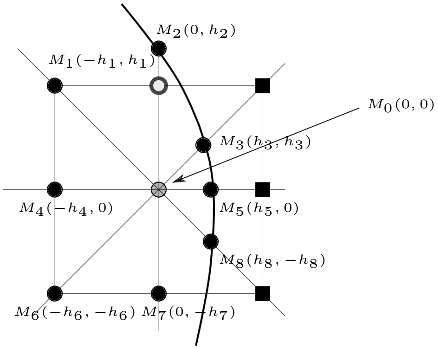


Figure: Embedding of an elliptical domain in a Cartesian grid. Calculated nodes : black circles. Exterior points : black squares. Edge Points: white circles.



[8] M. Ben-Artzi, I. Chorev, J.-P. Croisille and D. Fishelov, SINUM 2009.
[9] M. Ben-Artzi, J.-P. Croisille and D. Fishelov, BGSiam, 2017.

A Hermite-Lagrange interpolation problem in two dimensions

The sixth-order polynomial $P_{\mathbf{M_0}}(x,y)$ is of the form

$$P(x,y) = \sum_{i=1}^{19} a_i l_i(x,y), \tag{28}$$

$$\left\{ \begin{array}{c} l_1(x,y) = 1, \;\; l_2(x,y) = x, \;\; l_3(x,y) = x^2, \\ l_4(x,y) = x^3, \;\; l_5(x,y) = x^4, \;\; l_6(x,y) = x^5, \\ l_7(x,y) = y, \;\; l_8(x,y) = y^2, \;\; l_9(x,y) = y^3, \\ l_{10}(x,y) = y^4, \;\; l_{11}(x,y) = y^5, \;\; l_{12}(x,y) = xy, \\ l_{13}(x,y) = xy(x+y), \;\; l_{14}(x,y) = xy(x-y), \\ l_{15}(x,y) = xy(x+y)^2, \;\; l_{16}(x,y) = xy(x-y)^2, \;\; l_{17}(x,y) = xy(x+y)^3, \\ l_{18}(x,y) = xy(x-y)^3, \;\; l_{19}(x,y) = x^2y^2(x^2+y^2). \end{array} \right. \tag{29}$$

A High Order scheme for Irregular domains using 2D polynomials

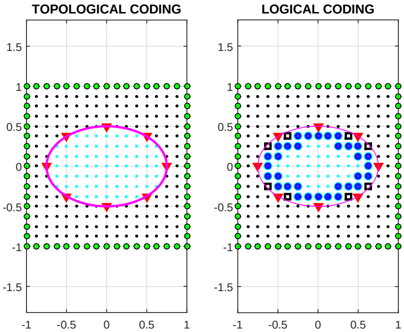


Figure: The ellipse $4x^2 + 16y^2 \leq 1$. Left: Exterior, boundary or internal. Right: Exterior, boundary, edge, interior regular or irregular calculated.

The mesh for Irregular domains

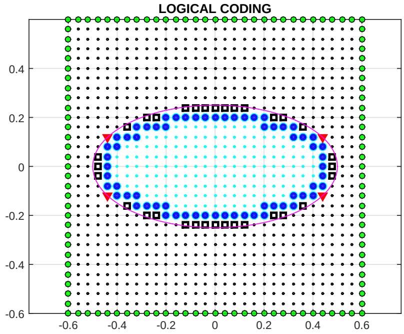


Figure: The embedded 30×30. Boundary points - red triangles. Edge points - black open squares. Irregular calculated - dark blue circles.

The Exact and Calculated solution for the ellipse

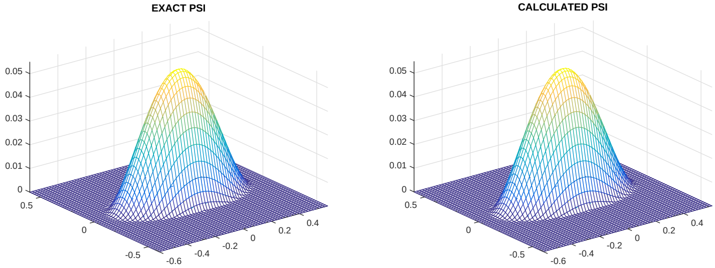


Figure: Ellipse embedded in a 60×60 Cartesian grid. NS for $\psi = (x^2 + 4y^2 - 0.25)^2$ in the ellipse $4x^2 + 16y^2 \leq 1$: Exact and calculated solutions at final time $t_f = 0.5$.

The errors for the ellipse

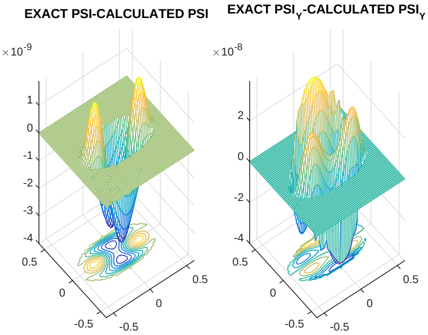
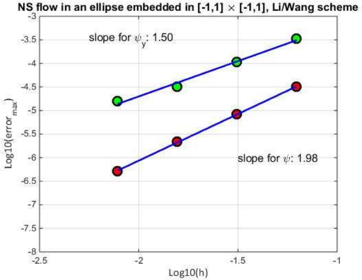
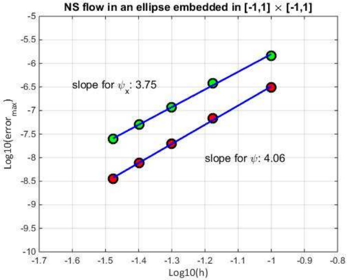


Figure: Error in ψ and ψ_y at $t_f = 0.5$, $\nu = 0.001$, 60×60 mesh for the ellipse $4x^2 + 16y^2 \leq 1$.

Rates of Convergence for the Ellipse



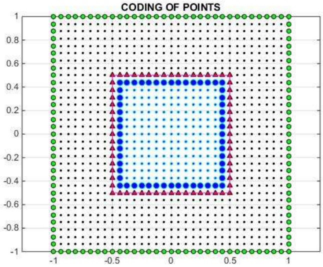
(a) Li and Wang scheme



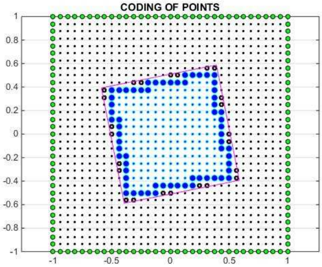
(b) Present scheme

Figure: Regression lines for the Li-Wang test case. Left: Li and Wang convergence rate with $N = 32, 64, 128, 256$. Right: Present scheme with $N = 20, 30, 40, 50, 60$. Note that the accuracy with $N = 20$ on the right is better than the accuracy with $N = 256$ on the left.

Consistency of the accuracy under rotation



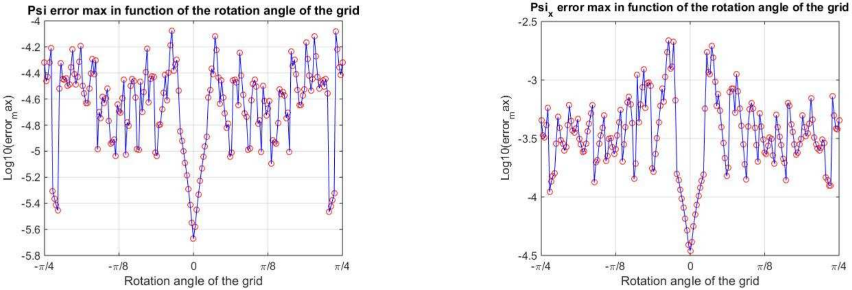
(a) Coding of Points $\theta = 0$



(b) Coding of Points $\theta = \pi/16$

Figure: Labeling of points in the square $[-0.5, 0.5]$ embedded in the computational square $[-1, 1] \times [-1, 1]$ after rotation. (a) at position $\theta = 0$, (b) $\theta = \pi/16$.

Consistency of the accuracy under rotation



(a) ψ error after rotation

(b) ψ_x error after rotation

Figure: Maximum error for the Navier-Stokes equation in the square $[-0.5, 0.5] \times [-0.5, 0.5]$. Computation for $\pi/4 + k\pi/360$ for all $k = 0, ..., 180$. Left: accuracy for $\psi(t, x, y)$ at final time on the grid k. Right: accuracy for $\psi_x(t, x, y)$ at final time on the grid k.

A High Order scheme for Irregular domains using 1D polynomials

Define a new coordinate system

$$\eta = (x+y)/\sqrt{2}, \quad \xi = (y-x)/\sqrt{2}.$$

Thus, $y = (\eta+\xi)/\sqrt{2}, \quad x = (\eta-\xi)/\sqrt{2}$. By the chain rule,

$$\begin{aligned} \psi_{\eta\eta\eta\eta} &= \tfrac{1}{4}(\psi_{xxxx} + 4\psi_{xxxy} + 6\psi_{xxyy} + 4\psi_{xyyy} + \psi_{yyyy}) \\ \psi_{\xi\xi\xi\xi} &= \tfrac{1}{4}(\psi_{xxxx} - 4\psi_{xxxy} + 6\psi_{xxyy} - 4\psi_{xyyy} + \psi_{yyyy}). \end{aligned} \tag{30}$$

Therefore, $2(\psi_{\eta\eta\eta\eta} + \psi_{\xi\xi\xi\xi}) = \psi_{xxxx} + 6\psi_{xxyy} + \psi_{yyyy}$. Thus,

$$\begin{aligned} \Delta^2\psi &= \psi_{xxxx} + 2\psi_{xxyy} + \psi_{yyyy} \\ &= \tfrac{2}{3}(\psi_{\eta\eta\eta\eta} + \psi_{\xi\xi\xi\xi} + \psi_{xxxx} + \psi_{yyyy}). \end{aligned} \tag{31}$$

A. Ditkowski, private communications

The discrete convective for an irregular element

The convective term is

$$C(\psi) = \nabla^{\perp}\psi \cdot \nabla\Delta\psi = -(\partial_y\psi)\cdot\partial_x(\Delta\psi) + (\partial_x\psi)\cdot\partial_y(\Delta\psi).$$

It may be written as

$$C(\psi) = -(\partial_y\psi)\cdot(\partial_{xxx}\psi + \partial_{xyy}\psi) + (\partial_x\psi)\cdot(\partial_{xxy}\psi + \partial_{yyy}\psi).$$

[10] D. Fishelov, Computers and Mathematics with Applications, 2017.

The discrete convective for an irregular element

For the mixed third-order derivatives we have.

$$\psi_{\eta\eta\eta} = \frac{1}{2\sqrt{2}}(\psi_{xxx} + 3\psi_{xxy} + 3\psi_{xyy} + \psi_{yyy}),$$
$$\psi_{\xi\xi\xi} = \frac{1}{2\sqrt{2}}(-\psi_{xxx} + 3\psi_{xxy} - 3\psi_{xyy} + \psi_{yyy}).$$

Therefore,

$$\psi_{xxy} = \frac{\sqrt{2}}{3}(\psi_{\eta\eta\eta} + \psi_{\xi\xi\xi}) - \frac{1}{3}\psi_{yyy},$$
$$\psi_{xyy} = \frac{\sqrt{2}}{3}(\psi_{\eta\eta\eta} - \psi_{\xi\xi\xi}) - \frac{1}{3}\psi_{xxx}.$$

The convective term may be written using only pure derivatives by

$$C(\psi) = -\psi_y \cdot \Big(\frac{2}{3}\psi_{xxx} + \frac{\sqrt{2}}{3}(\psi_{\eta\eta\eta} - \psi_{\xi\xi\xi})\Big) + \psi_x \cdot \Big(\frac{2}{3}\psi_{yyy} + \frac{\sqrt{2}}{3}(\psi_{\eta\eta\eta} + \psi_{\xi\xi\xi})\Big).$$

The truncation error for ∂_x, ∂_x^4 for an irregular element

Let $Q(x)$ be the following polynomial with interpolating data

$$Q(x) = a_0 + a_1(x - x_i) + a_2(x - x_i)^2 + a_3(x - x_i)^3 + a_4(x - x_i)^4,$$

$$\psi(x_{west}, y_j), \psi(x_i, y_j), \psi(x_{east}, y_j), \psi_x(x_{west}, y_j), \psi_x(x_{east}, y_j).$$

Then, the approximation $\psi_{x,i,j}$ to $\partial_x \psi_{i,j}$ has the form

$$\begin{aligned} &\psi_{x,i,j} + c_{x,p} \cdot \psi_x(x_{east}, y_j) + c_{x,m} \cdot \psi_x(x_{west}, y_j) \\ &= c_p \cdot \psi(x_{east}, y_j) - c_m \cdot \psi(x_{west}, y_j) - c \cdot \psi_{i,j}. \end{aligned} \tag{32}$$

The truncation errors for ψ_x and $\bar{\delta}_x^4$ for an irregular element satisfy

$$|(\psi_x)_{i,j} - \partial_x \psi| \leq Ch^4 \|\psi^{(5)}\|_{L^\infty},$$

$$|\bar{\delta}_x^4 \psi_{i,j} - \partial_x^4 \psi| \leq Ch \|\psi^{(5)}\|_{L^\infty}.$$

Numerical Results- Irregular Domains-Full Navier-Stokes

Intersection of two non-concentric circles

$$\Omega = \{(x,y) | (x-0.4)^2 + y^2 < 0.5\} \cap \{(x,y) | (x+0.4)^2 + y^2 < 0.5\} \quad (33)$$

$$\psi(x,y,t) = \frac{1}{64}(0.81 - (x^2+y^2)^2)e^{-t} \text{ in } \Omega$$

We resolve numerically

$$\begin{cases} \partial_t \Delta\psi + \nabla^{\perp}\psi \cdot \nabla\Delta\psi - \Delta^2\psi = f(x,y,t), \quad (x,y) \in \Omega \\ \psi(x,y,0) = \frac{1}{64}(0.81 - (x^2+y^2)^2), \quad (x,y) \in \Omega \\ \psi(x,y,t) = \frac{1}{64}(0.81 - (x^2+y^2)^2)e^{-t}, \quad (x,y) \in \partial\Omega \\ \frac{\partial\psi(x,y,t)}{\partial n} = \frac{1}{64}\frac{\partial((0.81-(x^2+y^2)^2)e^{-t}}{\partial n}, \quad (x,y) \in \partial\Omega. \end{cases} \quad (34)$$

mesh	11×11	Rate	21×21	Rate	41×41
e_2	5.8018E-09	3.87	3.9712E-10	3.86	2.7436E-11
e_∞	1.1809E-08	4.20	7.25789E-10	3.98	4.6122E-11
$(e_x)_2$	2.1158E-08	4.30	1.0708E-09	3.86	7.3503E-11
$(e_x)_\infty$	3.7714E-08	4.15	2.1361E-09	3.94	1.3377E-10

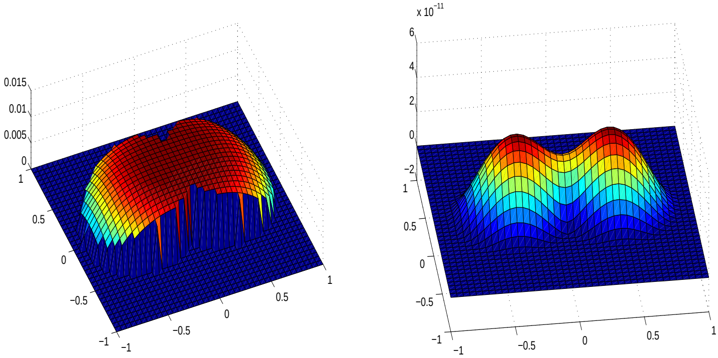


Figure: Left: Approximation for $\psi(x,y,t) = \frac{1}{64}(0.81 - (x^2+y^2)^2)e^{-t}$. Right: The error

$$\psi(x,y,t) = (1/64)e^{-t}((x^2+y^2)^2 + e^x\cos(y))$$

Our aim is to recover $\psi(x,y,t)$ from $f(x,y,t)$. Thus, we resolve numerically

$$\begin{cases} \partial_t \Delta\psi + \nabla^{\perp}\psi \cdot \nabla\Delta\psi - \Delta^2\psi = f(x,y,t), & (x,y) \in \Omega \\ \psi(x,y,0) = (1/64)((x^2+y^2)^2 + e^x\cos(y)), & (x,y) \in \Omega \\ \psi(x,y,t) = (1/64)e^{-t}((x^2+y^2)^2 + e^x\cos(y)), & (x,y) \in \partial\Omega \\ \frac{\partial\psi(x,y,t)}{\partial n} = \frac{\partial(1/64)e^{-t}((x^2+y^2)^2+e^x\cos(y))}{\partial n}, & (x,y) \in \partial\Omega. \end{cases} \quad (35)$$

mesh	11×11	Rate	21×21	Rate	41×41
e_2	3.0809E-08	4.02	1.8993E-09	4.33	9.4105E-11
e_∞	9.6878E-08	4.21	5.2525E-09	4.25	2.7563E-10
$(e_x)_2$	2.8732E-07	4.17	1.5968E-08	4.16	8.9395E-10
$(e_x)_\infty$	5.6380E-07	4.28	2.8971E-08	3.63	2.3323E-09

Table 10: Compact scheme for Navier-Stokes equation with exact solution: $\psi = (1/64)e^{-t}((x^2+y^2)^2 + e^x \cos(y))$ on Ω. We present e and e_x, the l_2 errors for the streamfunction and for $\partial_x \psi$. Here $\Delta t = 0.25h^2$ and $t = 0.16$.

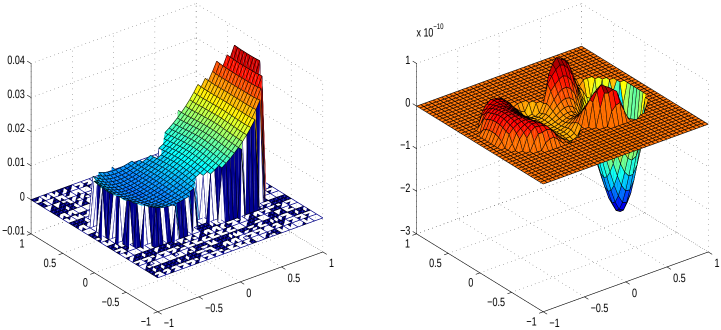


Figure: Left: Approximation for $\psi(x,y,t) = (1/64)e^{-t}((x^2+y^2)^2 + e^x\cos(y))$. Right: The error

Approximate biharmonic spectral problems in two dimensions

Let $\Omega \subset \mathbb{R}^2$. We consider the two following eigenproblems in Ω

Problem 1: The buckling plate problem

$$\Delta^2 \psi = -\lambda \Delta \psi, \quad \mathbf{x} \in \Omega. \tag{36}$$

Problem 2: The clamped plate problem

$$\Delta^2 \psi = \lambda \psi, \quad \mathbf{x} \in \Omega. \tag{37}$$

In each case, we want to calculate approximations of the $(\lambda_n, \psi_n(\mathbf{x}))$, $n \geq 1$, the eigenvalues of the problem which are ordered in ascending order.

Approximate biharmonic spectral problems for Problem 1 in the square

N	$\lambda_1(N)$ our scheme	$\lambda_1(N)$ (Brenner-Monk-Sun)
10	52.316494	55.4016
20	52.343018	53.2067
40	52.344588	52.5757
80	52.344685	52.4045

The value obtained by (Bjørstad and Tjøstheim) (1999) in the square is

$$\lambda_1 = 52.3446911684165445387053307503 65 \quad (38)$$

Approximate biharmonic spectral problems for Problem 1 in a square

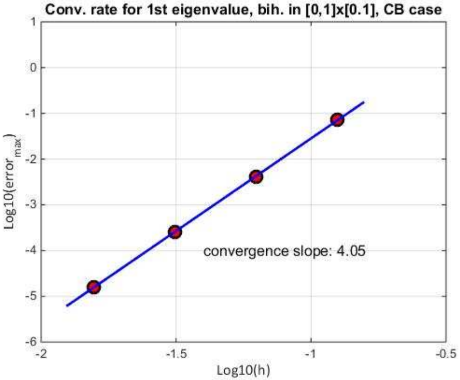


Figure: Convergence rate for the Problem 1.

Approximate biharmonic spectral problems for Problem 2 in the square

N	$\lambda_1(N)$ by (1)	$\lambda_1(N)$ (Brenner-Monk-Sun)
10	1295.434650	1377.1366
20	1294.973270	1318.5091
40	1294.436592	1301.3047
80	1294.934146	1296.5904

The value obtained by Bjørstad and Tjøstheim (1999) in the square is

$$\lambda_1 = 1294.9339795917128081703026479744...$$

Approximate biharmonic spectral problems for Problem 1 in a square

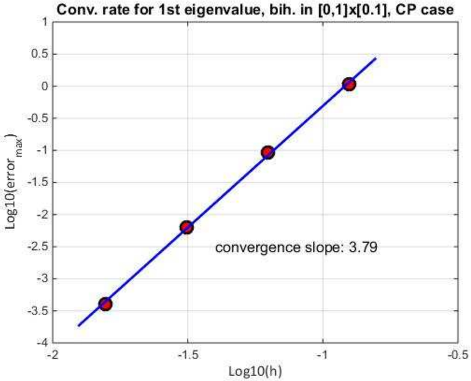


Figure: Convergence rate for the Problem 2.

Eigenfunctions for Problem 2 in a disc $x^2 + y^2 \leq 1$

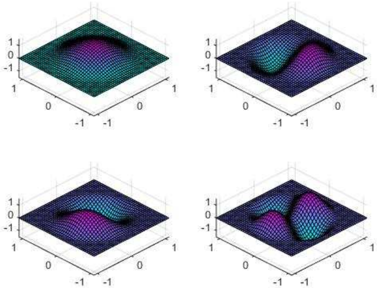


Figure: Eigenfunctions for λ_1, λ_2, λ_3, λ_4 for Problem 1 in the disc. The size of the grid is 40×40.

Eigenfunctions for Problem 2 in a disc $x^2 + y^2 \leq 1$

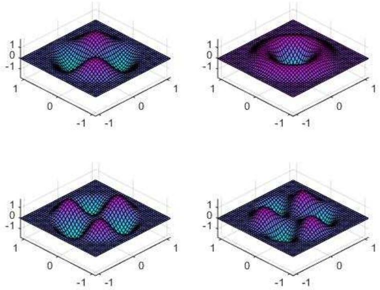


Figure: Eigenfunctions for λ_5, λ_6, λ_7, λ_8 for Problem 1 in the disc. The size of the grid is 40×40.

Eigenfunctions for Problem 2 in a disc $x^2 + y^2 \leq 1$

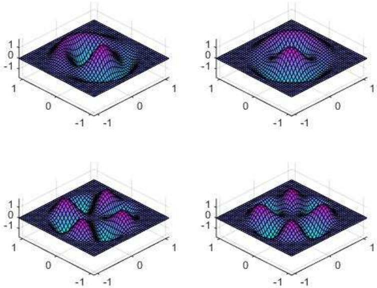


Figure: Eigenfunctions for λ_9, λ_{10}, λ_{11}, λ_{12} for Problem 2 in the disc $x^2 + y^2 \leq 1$. The size of the grid is 40×40.

Eigenfunctions for Problem 2 in a disc $x^2 + y^2 \leq 1$

N	$\lambda_1(N)$	$\lambda_2(N)$	$\lambda_3(N)$	$\lambda_4(N)$
10	0.1043056(3)	0.4510779(3)	0.4510779(3)	1.2105913(3)
20	0.1043621(3)	0.4519756(3)	0.4519756(3)	1.2159930(3)
40	0.1043630(3)	0.4520028(3)	0.4520028(3)	1.2163867(3)
80	0.1043631(3)	0.4520044(3)	0.4520044(3)	1.2164070(3)

Table: Disk $x^2 + y^2 \leq 1$ embedded in the square $[-1.1, 1.1] \times [-1, 1]$. Approximate value of the four smallest eigenvalues of the clampled plate eigenproblem (37) for $h = 1/10, 1/20, 1/40$ and $1/80$.

Eigenfunctions for Problem 2 in the ellipse $x^2/0.7^2 + y^2/1.3^2 \leq 1$

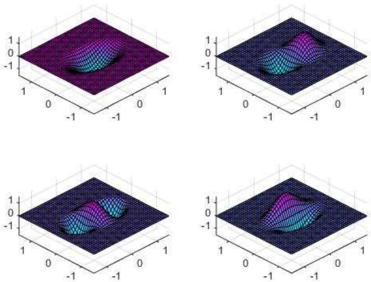


Figure: Eigenfunctions for λ_2, λ_3, λ_4, λ_5 for Problem 1 in the ellipse. The size of the grid is 40×40.

Eigenfunctions for Problem 2 in the ellipse $x^2/0.7^2 + y^2/1.3^2 \leq 1$

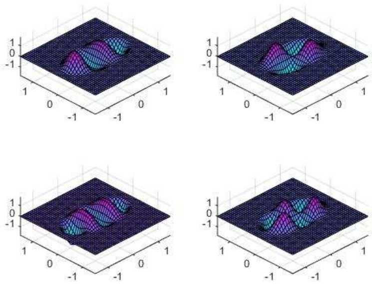


Figure: Eigenfunctions for λ_6, λ_7, λ_8, λ_9 for Problem 2 in the ellipse. The size of the grid is 40×40.

Eigenfunctions for Problem 2 in the ellipse $x^2/0.7^2 + y^2/1.3^2 \leq 1$

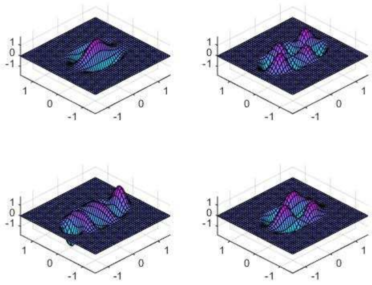


Figure: Eigenfunctions for λ_9, λ_{10}, λ_{11}, λ_{12} for Problem 2 in the ellipse $x^2/0.7^2 + y^2/1.3^2 \leq 1$. The size of the grid is 40×40.

Convergence of eigenvalues for Problem 2 in the ellipse

N	$\lambda_1(N)$	$\lambda_2(N)$	$\lambda_3(N)$	$\lambda_4(N)$
10	0.2031296(3)	0.4525390(3)	0.9525603(3)	1.2572786(3)
20	0.2038618(3)	0.4531441(3)	0.9561893(3)	1.2995270(3)
40	0.2038890(3)	0.4531487(3)	0.9564064(3)	1.3003232(3)
80	0.2038902(3)	0.4531510(3)	0.9564114(3)	1.3004021(3)

Table: Ellipse $x^2/0.7^2 + y^2/1.3^2 \leq 1$ embedded in the square $[-1.6, 1.6] \times [-1.6, 1.6]$. Approximate value of the four smallest eigenvalues of the clampled plate eigenproblem (37) for $h = 1/10, 1/20, 1/40$ and $1/80$.

Thanks for your attention!