Structure Preserving Numerical Methods for Hyperbolic Systems of Conservation and Balance Laws

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joint work with
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Systems of Balance Laws

$$U_t + f(U)_x + g(U)_y = S(U)$$

Examples:

- Gas dynamics with pipe-wall friction
- Euler equations with gravity/friction
- shallow water equations with Coriolis forces

Applications:

- astrophysical and atmospheric phenomena in many fields including supernova explosions
- (solar) climate modeling and weather forecasting

$$U_t + f(U)_x + g(U)_y = \frac{1}{\varepsilon} S(U)$$

Examples:

- low Mach number compressible flows
- low Froude number shallow water flows
- diffusive relaxation in kinetic models

Applications:

- various two-phase flows such as bubbles in water
- unmostly incompressible flows with regions of high compressibility such as underwater explosions
- atmospheric flows

Systems of Balance Laws

$$\boldsymbol{U}_t + \boldsymbol{f}(\boldsymbol{U})_x + \boldsymbol{g}(\boldsymbol{U})_y = \boldsymbol{S}(\boldsymbol{U}) \quad \text{or} \quad \boldsymbol{U}_t + \boldsymbol{f}(\boldsymbol{U})_x + \boldsymbol{g}(\boldsymbol{U})_y = \frac{1}{\varepsilon}\boldsymbol{S}(\boldsymbol{U})$$

- Challenges: certain structural properties of these hyperbolic problems (conservation or balance law, equilibrium state, positivity, assymptotic regimes, etc.) are essential in many applications;

- Goal: to design numerical methods that are not only consistent with the given PDEs, but
  - preserve the structural properties at the discrete level – **well-balanced numerical methods**
  - remain accurate and robust in certain asymptotic regimes of physical interest – **asymptotic preserving numerical methods**

[P. LeFloch; 2014]

Well-Balanced (WB) Methods

$$\boldsymbol{U}_t + \boldsymbol{f}(\boldsymbol{U})_x + \boldsymbol{g}(\boldsymbol{U})_y = \boldsymbol{S}(\boldsymbol{U})$$

- In many physical applications, solutions of the system are small perturbations of the steady states;
- These perturbations may be smaller than the size of the truncation error on a coarse grid;
- To overcome this difficulty, one can use very fine grid, but in many physically relevant situations, this may be unaffordable;

Goal:

- to design a well-balanced numerical method, that is, the method which is capable of exactly preserving some steady state solutions;
- perturbations of these solutions will be resolved on a coarse grid in a non-oscillatory way.

Asymptotic Preserving (AP) Methods

$$\boldsymbol{U}_t + \boldsymbol{f}(\boldsymbol{U})_x + \boldsymbol{g}(\boldsymbol{U})_y = \frac{1}{\varepsilon}\boldsymbol{S}(\boldsymbol{U})$$

- Solutions of many hyperbolic systemes reveal a multiscale character and thus their numerical resolution presence some major difficulties;
- Such problems are typically characterized by the occurence of a small parameter by $0 < \varepsilon \ll 1$;
- The solutions show a nonuniform behavior as $\varepsilon \to 0$;
- the type of the limiting solution is different in nature from that of the solutions for finite values of $\varepsilon > 0$.

Goal:

- asymptotic passage from one model to another should be preserved at the discrete level;
- for a fixed mesh size and time step, AP method should automatically transform into a stable discretization of the limitting model as $\varepsilon \to 0$.

Finite-Volume Methods – 1-D

$$\boldsymbol{U}_t + \boldsymbol{f}(\boldsymbol{U})_x = \boldsymbol{S} \quad \left(= \frac{1}{\varepsilon}\boldsymbol{S}\right)$$

- $\overline{\boldsymbol{U}}_k^n \approx \dfrac{1}{\Delta y} \displaystyle\int\limits_{C_k} \boldsymbol{U}(y,t^n)\,dy$: cell averages over $C_j := (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$

- Semi-discrete FV method:

$$\frac{d}{dt}\overline{\boldsymbol{U}}_j(t) = -\frac{\boldsymbol{\mathcal{F}}_{j+\frac{1}{2}}(t) - \boldsymbol{\mathcal{F}}_{j-\frac{1}{2}}(t)}{\Delta x} + \overline{\boldsymbol{S}}_j$$

$\boldsymbol{\mathcal{F}}_{j+\frac{1}{2}}(t)$: numerical fluxes

$\overline{\boldsymbol{S}}_j$: quadrature approximating the corresponding source terms

- Central-Upwind (CU) Scheme:

[Kurganov, Lin, Noelle, Petrova, Tadmor, et al.; 2000–2007]

$$\{\overline{\boldsymbol{U}}_j(t)\} \to \widetilde{\boldsymbol{U}}(\cdot,t) \to \left\{\boldsymbol{U}_j^{\mathrm{E,W}}(t)\right\} \to \left\{\boldsymbol{\mathcal{F}}_{j+\frac{1}{2}}(t)\right\} \to \{\overline{\boldsymbol{U}}_j(t+\Delta t)\}$$

(Discontinuous) piecewise-linear reconstruction:

$$\widetilde{\boldsymbol{U}}(y,t) := \overline{\boldsymbol{U}}_j(t) + (\boldsymbol{U}_x)_j(x-x_j), \quad x \in C_j$$

It is conservative, second-order accurate, and non-oscillatory provided the slopes, $\{(\boldsymbol{U}_y)_k\}$, are computed by a nonlinear limiter

Example — Generalized Minmod Limiter

$$(\boldsymbol{U}_y)_j = \mathrm{minmod}\left(\theta\frac{\overline{\boldsymbol{U}}_j - \overline{\boldsymbol{U}}_{j-1}}{\Delta x}, \frac{\overline{\boldsymbol{U}}_{j+1} - \overline{\boldsymbol{U}}_{j-1}}{2\Delta x}, \theta\frac{\overline{\boldsymbol{U}}_{j+1} - \overline{\boldsymbol{U}}_j}{\Delta x}\right)$$

where

$$\mathrm{minmod}(z_1, z_2, ...) := \begin{cases} \min_j\{z_j\}, & \text{if } z_j > 0 \ \ \forall j, \\ \max_j\{z_j\}, & \text{if } z_j < 0 \ \ \forall j, \\ 0, & \text{otherwise,} \end{cases}$$

and $\theta \in [1,2]$ is a constant

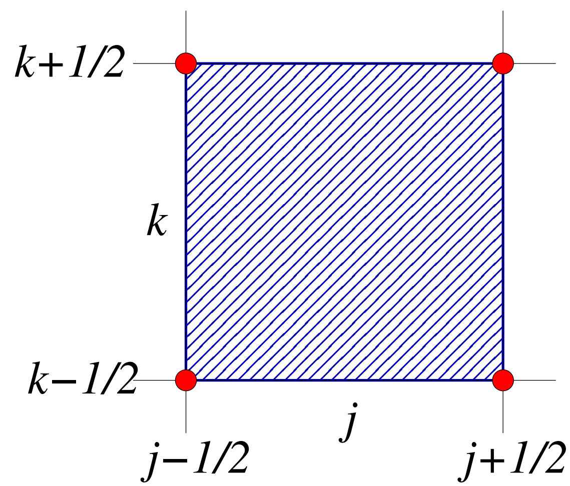
$$\{\overline{\boldsymbol{U}}_j(t)\} \to \widetilde{\boldsymbol{U}}(\cdot,t) \to \left\{\boldsymbol{U}_j^{\mathrm{E,W}}(t)\right\} \to \left\{\boldsymbol{\mathcal{F}}_{j+\frac{1}{2}}(t)\right\} \to \{\overline{\boldsymbol{U}}_j(t+\Delta t)\}$$

$\boldsymbol{U}_j^{\mathrm{E}}$ and $\boldsymbol{U}_j^{\mathrm{W}}$ are the point values at $x_{j+\frac{1}{2}}$ and $x_{j-\frac{1}{2}}$:

$$\widetilde{\boldsymbol{U}}(y,t) = \overline{\boldsymbol{U}}_j + (\boldsymbol{U}_x)_j(x - x_j), \quad x \in C_j$$

$$\boldsymbol{U}_j^{\mathrm{E}} := \overline{\boldsymbol{U}}_j + \frac{\Delta x}{2}(\boldsymbol{U}_x)_j$$

$$\boldsymbol{U}_j^{\mathrm{W}} := \overline{\boldsymbol{U}}_j - \frac{\Delta x}{2}(\boldsymbol{U}_x)_j$$



$$\{\overline{\boldsymbol{U}}_j(t)\} \to \widetilde{\boldsymbol{U}}(\cdot,t) \to \left\{\boldsymbol{U}_j^{\mathrm{E,W}}(t)\right\} \to \left\{\boldsymbol{\mathcal{F}}_{j+\frac{1}{2}}(t)\right\} \to \{\overline{\boldsymbol{U}}_j(t+\Delta t)\}$$

$$\frac{d}{dt}\overline{\boldsymbol{U}}_j = -\frac{\boldsymbol{\mathcal{F}}_{j+\frac{1}{2}} - \boldsymbol{\mathcal{F}}_{j-\frac{1}{2}}}{\Delta x} + \overline{\boldsymbol{S}}_j$$

where

$$\boldsymbol{\mathcal{F}}_{j+\frac{1}{2}} = \frac{a^+_{j+\frac{1}{2}}\boldsymbol{f}(\boldsymbol{U}_j^{\mathrm{E}}) - a^-_{j+\frac{1}{2}}\boldsymbol{f}(\boldsymbol{U}_{j+1}^{\mathrm{W}})}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} + \alpha_{j+\frac{1}{2}}\left(\boldsymbol{U}_{j+1}^{\mathrm{W}} - \boldsymbol{U}_j^{\mathrm{E}}\right)$$

$$\alpha_{j+\frac{1}{2}} = \frac{a^+_{j+\frac{1}{2}}a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}$$

$$a^+_{j+\frac{1}{2}} = \max\left\{\lambda(\boldsymbol{U}_j^{\mathrm{E}}), \lambda(\boldsymbol{U}_{j+1}^{\mathrm{W}}), 0\right\}, \quad a^-_{j+\frac{1}{2}} = \min\left\{\lambda(\boldsymbol{U}_j^{\mathrm{E}}), \lambda(\boldsymbol{U}_{j+1}^{\mathrm{W}}), 0\right\}$$

2-D extension is dimension-by-dimension

Non Well-Balanced Property – Example

$$\begin{cases} \rho_t + q_x = 0, \\ q_t + f_2(\rho, q)_x = -s(\rho, q) \end{cases}$$

For steady-state solution: $q = \text{Const}$ and $\rho = \rho(x)$

Implementing the CU scheme results in

$$\frac{d\bar{\rho}_j}{dt} = -\frac{1}{\Delta x}\left[\cancel{\frac{a^+_{j+\frac{1}{2}} q^{\mathrm{E}}_j - a^-_{j+\frac{1}{2}} q^{\mathrm{W}}_{j+1}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}} + \alpha_{j+\frac{1}{2}}(\rho^{\mathrm{W}}_{j+1} - \rho^{\mathrm{E}}_j)\right.$$

$$\left. - \cancel{\frac{a^+_{j-\frac{1}{2}} q^{\mathrm{E}}_{j-1} - a^-_{j-\frac{1}{2}} q^{\mathrm{W}}_j}{a^+_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}}} + \alpha_{j-\frac{1}{2}}(\rho^{\mathrm{W}}_j - \rho^{\mathrm{E}}_{j-1})\right] \neq 0$$

- The steady state would not be preserved at the discrete level;
- This would also true for the first-order version of the scheme;
- For smooth solutions, the balance error is expected to be of order $(\Delta x)^2$, but a coarse grid solution may contain large spurious waves.

Well-Balanced Methods

“Balance is not something you find, it’s something you create”

1-D 2×2 Systems of Balance Laws

$$\begin{cases} \rho_t + f_1(\rho, q)_x = 0, \\ q_t + f_2(\rho, q)_x = -s(\rho, q), \end{cases}$$

Steady state solution:

$$f_1(\rho, q)_x \equiv 0, \quad f_2(\rho, q)_x + s(\rho, q) \equiv 0$$

or

$$\left. \begin{aligned} K &:= f_1(\rho, q) \equiv \mathrm{Const}, \\ L &:= f_2(\rho, q) + \int^x s(\rho, q) d\xi \equiv \mathrm{Const} \end{aligned} \right. \qquad \forall x, t$$

Numerical Challenges : to exactly balance the flux and source terms, i.e., to exactly preserve the steady states.

How to design a well-balanced scheme?

Well-Balanced Scheme

$$\begin{cases} \rho_t + f_1(\rho, q)_x = 0, \\ q_t + f_2(\rho, q)_x = -s(\rho, q) \end{cases}$$

- Incorporate the source term into the flux:

$$\begin{cases} \rho_t + f_1(\rho, q)_x = 0, \\ q_t + \left(f_2(\rho, q)_x + R\right)_x = 0, \end{cases} \qquad R := \int^x s(\rho, q)\,d\xi$$

- Rewrite

$$\begin{cases} \rho_t + K_x = 0, \\ q_t + L_x = 0 \end{cases}$$

where

$$K := f_1(\rho, q), \qquad L := f_2(\rho, q)_x + R$$

- Define

conservative variables $\boldsymbol{U} = (\rho, q)^T$

equilibrium variables $\mathbf{W} := (K, L)^T$

Well-Balanced Scheme

$$\boldsymbol{U}_t + \boldsymbol{f}(\boldsymbol{U})_x = \boldsymbol{0}$$

$$\boldsymbol{U} = \begin{pmatrix} \rho \\ q \end{pmatrix}, \quad \boldsymbol{f}(\boldsymbol{U}) = \boldsymbol{W} := \begin{pmatrix} K \\ L \end{pmatrix}$$

Semi-discrete FV method:

$$\frac{d}{dt}\overline{\boldsymbol{U}}_j(t) = -\frac{\boldsymbol{\mathcal{F}}_{j+\frac{1}{2}}(t) - \boldsymbol{\mathcal{F}}_{j-\frac{1}{2}}(t)}{\Delta x}$$

Two major modifications:

- Well-balanced reconstruction – *performed on the equilibrium rather than conservative variables*:

$$\{\overline{\boldsymbol{U}}_j(t)\} \to \widetilde{\boldsymbol{U}}(\cdot,t) \to \left\{\mathbf{W}_j^{\mathrm{E,W}}(t)\right\} \to \left\{\boldsymbol{U}_j^{\mathrm{E,W}}(t)\right\} \to \left\{\boldsymbol{\mathcal{F}}_{j+\frac{1}{2}}(t)\right\} \to \{\overline{\boldsymbol{U}}_j(t+\Delta t)\}$$

- Well-balanced evolution

Well-Balanced Reconstruction

Given: $\overline{\boldsymbol{U}}_j(t) = (\overline{\rho}_j, \overline{q}_j)^T$ – cell averages

Need: $\mathbf{W}_j^{\mathrm{E,W}} = (K_j^{\mathrm{E,W}}, L_j^{\mathrm{E,W}})^T$ – point values, where

$$K := f_1(\rho, q), \quad L := f_2(\rho, q)_x + R, \quad R := \int^x s(\rho, q)d\xi$$

- Compute $R_j = \int^{x_j} s(\rho, q)d\xi$ by the midpoint quadrature rule and using the following recursive relation:

$$R_{1/2} \equiv 0, \quad R_j = \frac{1}{2}(R_{j-\frac{1}{2}} + R_{j+\frac{1}{2}}),$$
$$R_{j+\frac{1}{2}} = R(x_{j+\frac{1}{2}}) = R_{j-\frac{1}{2}} + \Delta x\, s(x_j, \bar{\rho}_j, \bar{q}_j)$$

- Compute the point values of K and L at x_j from the cell averages, $\bar{\rho}_j$ and $\bar{q}_j$:

$$K_j = f_1(\bar{\rho}_j, \bar{q}_j), \qquad L_j = f_2(\bar{\rho}_j, \bar{q}_j) + R_j$$

Well-Balanced Reconstruction

- Apply the minmod reconstruction procedure to $\{K_j, L_j\}$ and obtain the point values at the cell interfaces:

$$K_j^{\mathrm{E}} = K_j + \frac{\Delta x}{2}(K_x)_j, \quad L_j^{\mathrm{E}} = L_j + \frac{\Delta x}{2}(L_x)_j,$$
$$K_j^{\mathrm{W}} = K_j - \frac{\Delta x}{2}(K_x)_j, \quad L_j^{\mathrm{W}} = L_j - \frac{\Delta x}{2}(L_x)_j$$

- Finally, equipped with the values of $K_j^{\mathrm{E,W}}$, $L_j^{\mathrm{E,W}}$ and $R_{j\pm\frac{1}{2}}$, solve

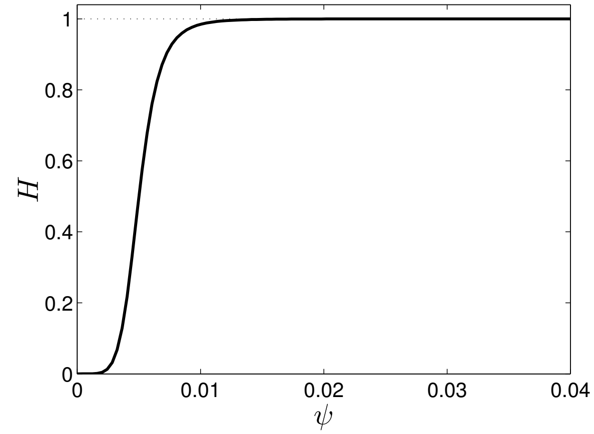
$$K_j^{\mathrm{E}} = f_1(\rho_j^{\mathrm{E}}, q_j^{\mathrm{E}}), \qquad L_j^{\mathrm{E}} = f_2(\rho_j^{\mathrm{E}}, q_j^{\mathrm{E}}) + R_{j+\frac{1}{2}},$$
$$K_j^{\mathrm{W}} = f_1(\rho_j^{\mathrm{W}}, q_j^{\mathrm{W}}), \quad L_j^{\mathrm{W}} = f_2(\rho_j^{\mathrm{W}}, q_j^{\mathrm{W}}) + R_{j-\frac{1}{2}}$$

for $\boldsymbol{U}_j^{\mathrm{E,W}} = (\rho_j^{\mathrm{E,W}}, q_j^{\mathrm{E,W}})^T$.

Well-Balanced Evolution

$$\frac{d}{dt}\overline{\boldsymbol{U}}_j = -\frac{\boldsymbol{\mathcal{F}}_{j+\frac{1}{2}} - \boldsymbol{\mathcal{F}}_{j-\frac{1}{2}}}{\Delta x}$$

where



$$\mathcal{F}^{(1)}_{j+\frac{1}{2}} = \frac{a^+_{j+\frac{1}{2}} K^{\mathrm{E}}_j - a^-_{j+\frac{1}{2}} K^{\mathrm{W}}_{j+1}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}$$

$$+ \alpha_{j+\frac{1}{2}}(\rho^{\mathrm{W}}_{j+1} - \rho^{\mathrm{E}}_j)\, \mathcal{H}\left(\frac{|K_{j+1} - K_j|}{\Delta x} \cdot \frac{|\Omega|}{\max_j} K_j, K_{j+1}\}\right),$$

$$\mathcal{F}^{(2)}_{j+\frac{1}{2}} = \frac{a^+_{j+\frac{1}{2}} L^{\mathrm{E}}_j - a^-_{j+\frac{1}{2}} L^{\mathrm{W}}_{j+1}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}$$

$$+ \alpha_{j+\frac{1}{2}}(q^{\mathrm{W}}_{j+1} - q^{\mathrm{E}}_j)\, \mathcal{H}\left(\frac{|L_{j+1} - L_j|}{\Delta x} \cdot \frac{|\Omega|}{\max_j\{L_j, L_{j+1}\}}\right),$$

Proof of the Well-Balanced Property

Theorem. *The central-upwind semi-discrete schemes coupled with the well-balanced reconstruction and evolution is well-balanced in the sense that it preserves the corresponding steady states exactly.*

Example – Gas dynamics with pipe-wall friction

$$\begin{cases} \rho_t + q_x = 0, \\ q_t + \left(c^2\rho + \dfrac{q^2}{\rho}\right)_x = -\mu\dfrac{q}{\rho}|q|, \end{cases}$$

- $\rho(x,t)$ is the density of the fluid
- $u(x,t)$ is the velocity of the fluid
- $q(x,t)$ is the momentum
- $\mu > 0$ is the friction coefficient (divided by the pipe cross section)
- $c > 0$ is the speed of sound

Equilibrium variables:

$$K(x,t) = q(x,t) \quad L(x,t) = \left(c^2\rho + \frac{q^2}{\rho}\right)(x,t) + R(x,t),$$

$$R(x,t) = \int^x \mu\frac{q(\xi,t)}{\rho(\xi,t)}|q(\xi,t)|d\xi$$

Steady states: $K \equiv \mathrm{Const}, \quad L \equiv \mathrm{Const}$

Numerical Tests

- Steady state initial data:

$$K(x,0) = q(x,0) = K^* = 0.15 \quad \text{and} \quad L(x,0) = L^* = 0.4,$$

in a single pipe $x \in [0,1]$

- Perturbed initial data:

$$K(x,0) = K^* + \eta e^{-100(x-0.5)^2}, \quad L(x,0) = L^* = 0.4, \quad \eta > 0$$

in a single pipe $x \in [0,1]$

We compare the WB and NWB methods ...

Numerical Test – Steady state initial data

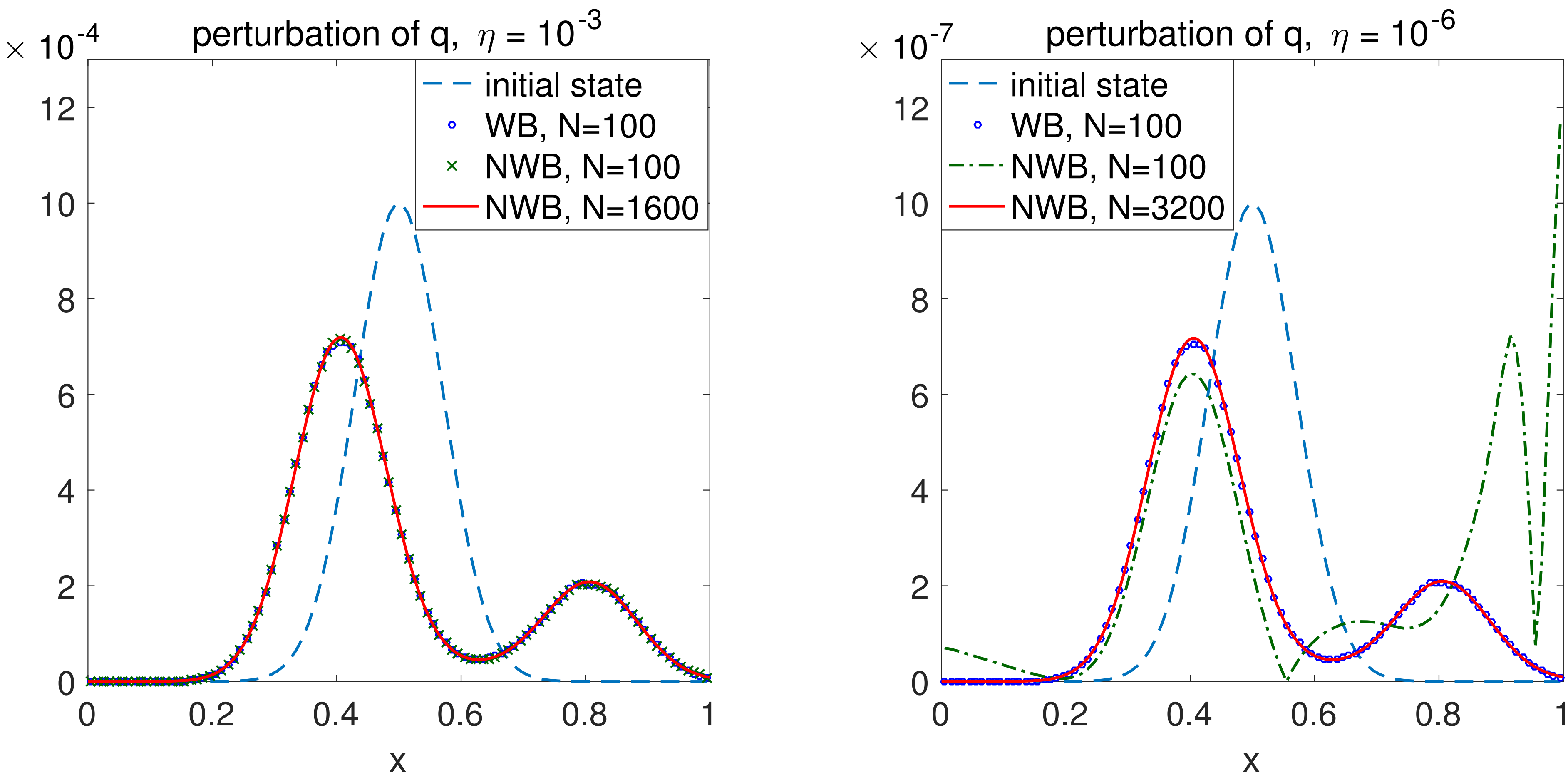
WB:

N	K	L
100	1.94E-18	7.77E-18
200	9.71E-19	9.71E-18
400	1.66E-18	9.57E-18
800	2.18E-18	1.18E-17

WB:

N	K	rate	L	rate
100	1.29E-06	-	8.81E-07	-
200	3.30E-07	1.9668	2.25E-07	1.9692
400	8.34E-08	1.9843	5.69E-08	1.9834
800	2.09E-08	1.9965	1.43E-08	1.9924

Numerical Test – Perturbed initial data



Euler Equations with Gravity

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0 \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = -\rho\phi_x \\ (\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = -\rho\phi_y \\ E_t + (u(E+p))_x + (v(E+p))_y = -\rho(u\phi_x + v\phi_y) \end{cases}$$

- ρ is the density
- u, v are the x- and y-velocities
- E is the total energy
- p is the pressure; $E = \dfrac{p}{\gamma - 1} + \dfrac{\rho}{2}(u^2 + v^2)$
- ϕ is the gravitational potential

Euler Equations with Gravity

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0 \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = -\rho\phi_x \\ (\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = -\rho\phi_y \\ E_t + (u(E+p))_x + (v(E+p))_y = -\rho(u\phi_x + v\phi_y) \end{cases}$$

Multiply the first (density) equation by ϕ and add to the last (energy) equation to obtain ...

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0 \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = -\rho\phi_x \\ (\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = -\rho\phi_y \\ (E + \rho\phi)_t + (u(E+\rho\phi+p))_x + (v(E+\rho\phi+p))_y = 0 \end{cases}$$

Steady States

$$\begin{cases} \cancel{\rho_t} + (\rho u)_x + (\rho v)_y = 0 \\ \cancel{(\rho u)_t} + (\rho u^2 + p)_x + (\rho uv)_y = -\rho\phi_x \\ \cancel{(\rho v)_t} + (\rho uv)_x + (\rho v^2 + p)_y = -\rho\phi_y \\ \cancel{(E + \rho\phi)_t} + (u(E + \rho\phi + p))_x + (v(E + \rho\phi + p))_y = 0 \end{cases}$$

Plays an important role in modeling model astrophysical and atmospheric phenomena in many fields including supernova explosions, (solar) climate modeling and weather forecasting

Steady state solution:

$$u \equiv 0, \ v \equiv 0, \ K_x = p_x + \rho\phi_x \equiv 0, \ L_y = p_y + \rho\phi_y \equiv 0$$

$$K := p + Q, \quad Q(x,y,t) := \int^x \rho(\xi,y,t)\phi_x(\xi,y)\,d\xi$$

$$L := p + R, \quad R(x,y,t) := \int^y \rho(x,\eta,t)\phi_y(x,\eta)\,d\eta$$

2-D Well-Balanced Scheme

- Incorporate the source term into the flux:

$$K := p + Q, \quad Q(x,y,t) := \int^{y} \rho(\xi,y,t)\phi_x(\xi,y), d\xi$$

$$L := p + R, \quad R(x,y,t) := \int^{y} \rho(x,\eta,t)\phi_y(x,\eta), d\eta$$

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E+\rho\phi \end{pmatrix}_t + \boxed{\begin{pmatrix} \rho u \\ \rho u^2 + K \\ \rho u v \\ u(E+\rho\phi+p) \end{pmatrix}_x + \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + L \\ v(E+\rho\phi+p) \end{pmatrix}_y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}$$

- Define

  conservative variables: $\boldsymbol{U} := (\rho, \rho u, \rho v, E)^T$

  equilibrium variables: $\boldsymbol{W} := (\rho, K, L, E+\rho\phi)^T$

- Solve by the well-balanced scheme ...

Well-Balanced Scheme

- Define

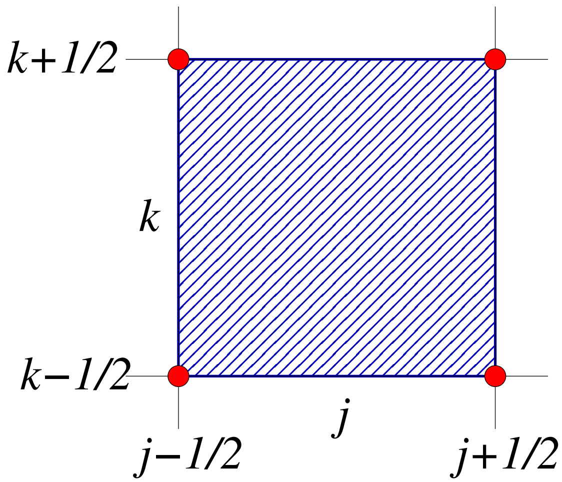
  conservative variables: $\boldsymbol{U} := (h, hu, hv)^T$

  equilibrium variables: $\boldsymbol{W} := (u, v, K, L)^T$

  fluxes in the x- and y-directions: $\boldsymbol{f}(\boldsymbol{U}, B)$ and $\boldsymbol{g}(\boldsymbol{U}, B)$

- Assume that at time t the cell averages are available

$$\overline{\boldsymbol{U}}_{j,k}(t) := \frac{1}{\Delta x \Delta y} \iint\limits_{C_{j,k}} \boldsymbol{U}(x, y, t)\, dxdy,$$



- Solve by the well-balanced scheme

$$\{\overline{\boldsymbol{U}}_{j,k}(t)\} \to \widetilde{\boldsymbol{U}}(\cdot, t) \to \left\{\mathbf{W}_{j,k}^{\mathrm{E,W,N,S}}(t)\right\} \to \left\{\boldsymbol{U}_{j,k}^{\mathrm{E,W,N,S}}(t)\right\}$$

$$\to \left\{\boldsymbol{\mathcal{F}}_{j+\frac{1}{2},k}(t), \boldsymbol{\mathcal{G}}_{j,k+\frac{1}{2}}(t)\right\} \to \{\overline{\boldsymbol{U}}_{j,k}(t + \Delta t)\}$$

Example — 2-D Isothermal Equilibrium Solution

[Xing, Shu; 2013]

- The ideal gas with $\gamma = 1.4$; domain $[0,1] \times [0,1]$
- The gravitational force is $\phi_y = \mathrm{g} = 1$
- The steady-state initial conditions are

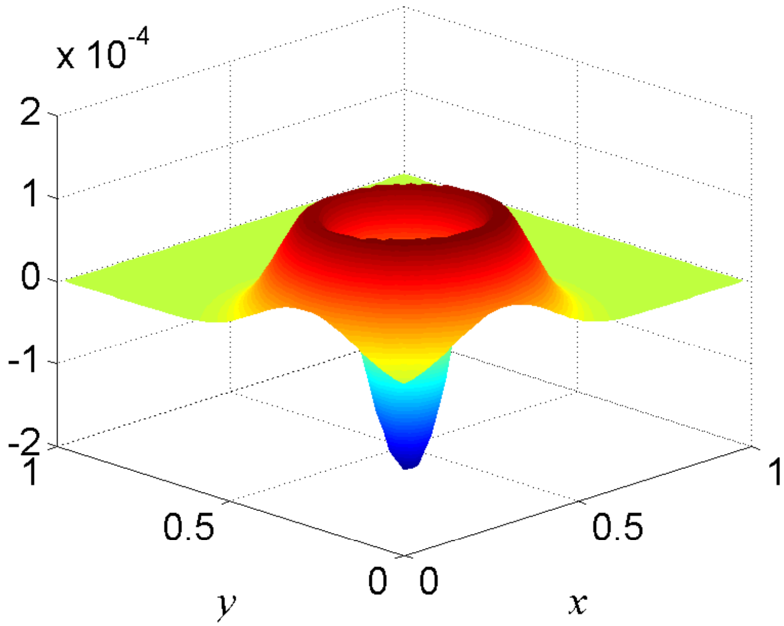
$$\rho(x,y,0) = 1.21e^{-1.21y}, \quad p(x,y,0) = e^{-1.21y}, \quad u(x,y,0) \equiv v(x,y,0) \equiv 0$$

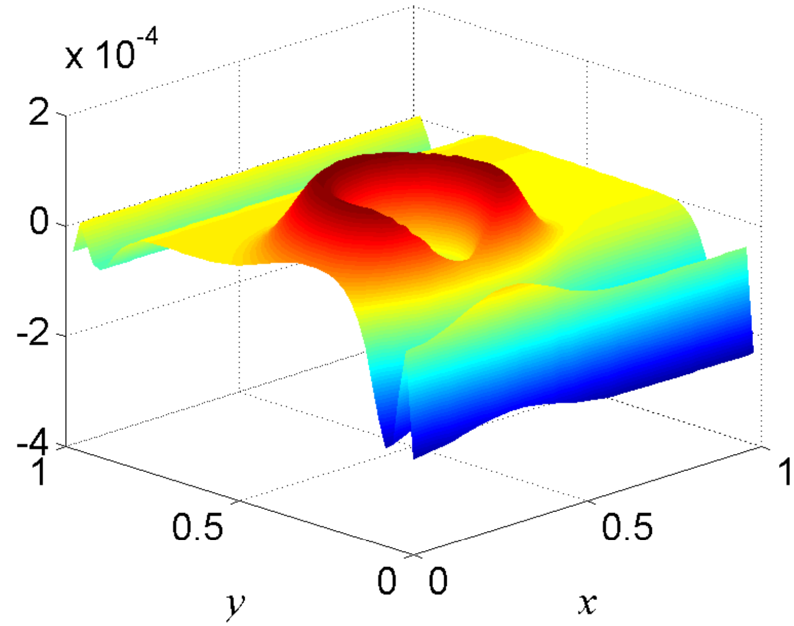
- Solid wall boundary conditions imposed at the edges of the unit square

Perturbation

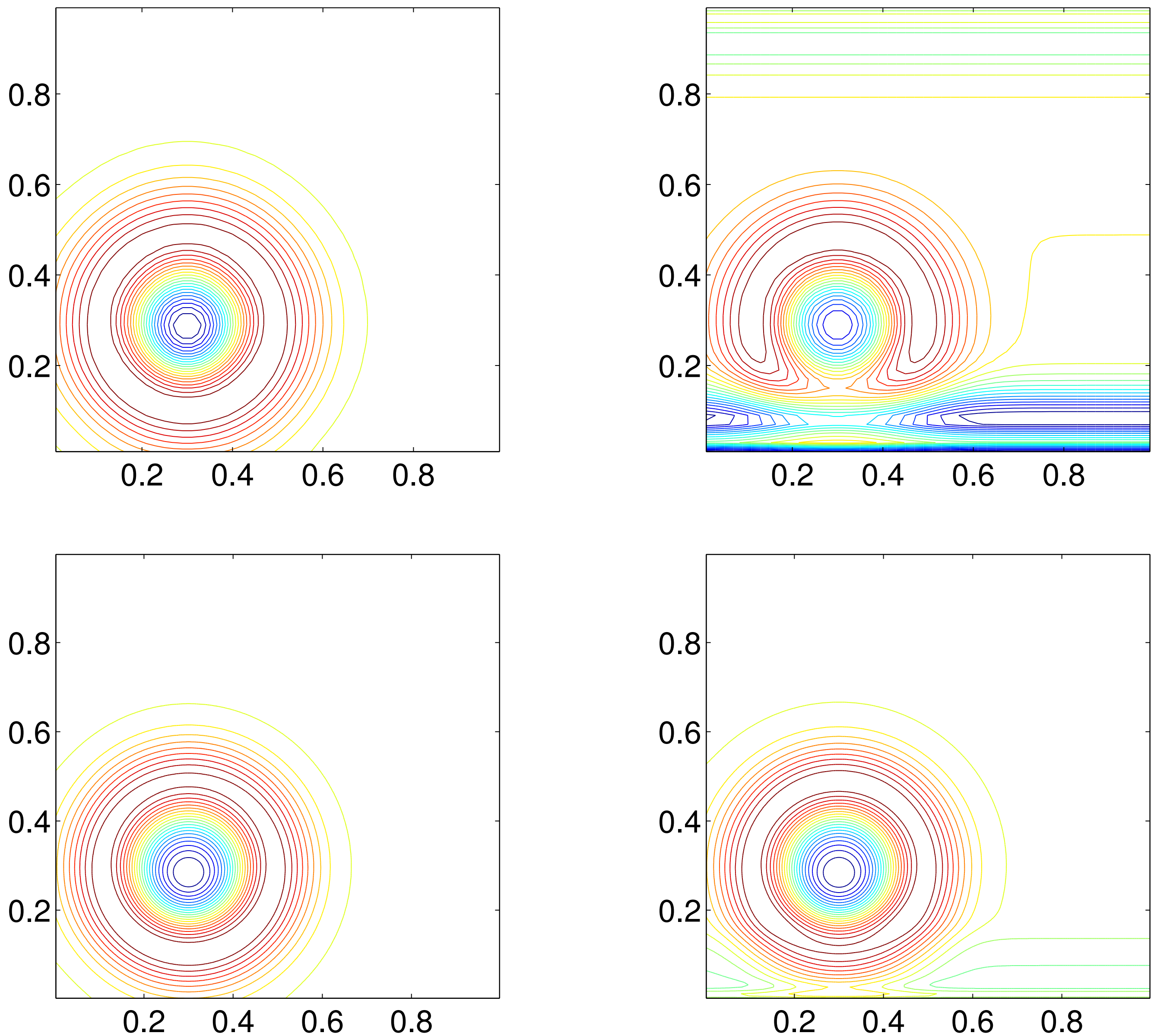
A small initial pressure perturbation:

$$p(x,y,0) = e^{-1.21y} + \eta e^{-121\left((x-0.3)^2+(y-0.3)^2\right)}, \quad \eta = 10^{-3}$$





50×50



WB : $50 \times 50,\ 200 \times 200$ NWB : $50 \times 50,\ 200 \times 200$

Shallow Water System with Coriolis Force

$$
\begin{cases}
h_t + (hu)_x + (hv)_y = 0 \\
(hu)_t + \left(hu^2 + \dfrac{g}{2}h^2\right)_x + (huv)_y = -ghB_x + fhv \\
(hv)_t + (huv)_x + \left(hv^2 + \dfrac{g}{2}h^2\right)_x = -ghB_y - fhu
\end{cases}
$$

- h: water height
- u, v: fluid velocity
- g: gravitational constant
- $B \equiv 0$ – bottom topography
- $f = 1/\varepsilon$ – Coriolis parameter

Dimensional Analysis

Introduce

$$\widehat{x} := \frac{x}{\ell_0}, \quad \widehat{y} := \frac{y}{\ell_0}, \quad \widehat{h} := \frac{h}{h_0}, \quad \widehat{u} := \frac{u}{w_0}, \quad \widehat{v} := \frac{v}{w_0}.$$

Substituting them into the SWE and dropping the hats in the notations, we obtain the dimensionless form:

$$\begin{cases} h_t + (hu)_x + (hv)_y = 0, \\ (hu)_t + \left(hu^2 + \dfrac{1}{\varepsilon^2}\dfrac{h^2}{2}\right)_x + (huv)_y = \dfrac{1}{\varepsilon}hv, \\ (hv)_t + (huv)_x + \left(hv^2 + \dfrac{1}{\varepsilon^2}\dfrac{h^2}{2}\right)_y = -\dfrac{1}{\varepsilon}hu, \end{cases}$$

in which

$$\mathrm{Fr} := \frac{w_0}{\sqrt{gh_0}} = \varepsilon$$

is the reference Froude number

Explicit Discretization

Eigenvalues of the flux Jacobian:

$$\left\{u \pm \frac{1}{\varepsilon}\sqrt{h},\, u\right\} \quad \text{and} \quad \left\{v \pm \frac{1}{\varepsilon}\sqrt{h},\, v\right\}$$

This leads to the CFL condition

$$\Delta t_{\text{expl}} \leq \nu \cdot \min\left(\frac{\Delta x}{\max\limits_{u,h}\left\{|u| + \frac{1}{\varepsilon}\sqrt{h}\right\}}, \frac{\Delta y}{\max\limits_{v,h}\left\{|v| + \frac{1}{\varepsilon}\sqrt{h}\right\}}\right) = \mathcal{O}(\varepsilon\Delta_{\min}).$$

where $\Delta_{\min} := \min(\Delta x, \Delta y)$

- $0 < \nu \leq 1$ is the CFL number
- Numerical diffusion: $\mathcal{O}(\lambda_{max}\Delta x) = \mathcal{O}(\varepsilon^{-1}\Delta x)$.
- We must choose $\Delta x \approx \varepsilon$ to control numerical diffusion and the stability condition becomes

$$\Delta t = \mathcal{O}(\varepsilon^2)$$

Low Froude Number Flows

Low Froude number regime $(0 < \varepsilon \ll 1) \Longrightarrow$ very large propagation speeds

Explicit methods:

- very restrictive time and space dicretization steps, typically proportional to ε due to the CFL condition;
- too computationally expensive and typically impractical.

Implicit schemes:

- uniformly stable for $0 < \varepsilon < 1$;
- may be inconsistent with the limit problem;
- may provide a wrong solution in the zero Froude number limit.

Goal: to design robust numerical algorithms, whose accuracy and efficiency is independent of ε

Asymptotic Perserving Methods

Asymptotic-Preserving (AP) Methods

Introduced in [Klar; 1998, Jin; 1999], see also [Jin, Levermore; 1991],

[Golse, Jin, Levermore; 1999].

Idea:

- asymptotic passage from one model to another should be preserved at the discrete level;
- for a fixed mesh size and time step, AP method should automatically transform into a stable discretization of the limitting model as $\varepsilon \to 0$.

$$\begin{tikzcd}
P^{\varepsilon,h} \arrow[r, "h \to 0"] \arrow[d, "\varepsilon \to 0"] & P^{\varepsilon} \arrow[d, "\varepsilon \to 0"] \\
P^{0,h} \arrow[r, "h \to 0"] & P^{0}
\end{tikzcd}$$

Hyperbolic Flux Splitting

Key idea: Split the stiff pressure term [Haack, Jin, Liu; 2012]

$$\begin{cases} h_t + \alpha(hu)_x + \alpha(hv)_y + (1-\alpha)(hu)_x + (1-\alpha)(hv)_y = 0, \\ (hu)_t + \left(hu^2 + \dfrac{\frac{1}{2}h^2 - a(t)h}{\varepsilon^2} \right)_x + (huv)_y + \dfrac{a(t)}{\varepsilon^2} h_x = \dfrac{1}{\varepsilon} hv, \\ (hv)_t + (huv)_x + \left(hv^2 + \dfrac{\frac{1}{2}h^2 - a(t)h}{\varepsilon^2} \right)_y + \dfrac{a(t)}{\varepsilon^2} h_y = -\dfrac{1}{\varepsilon} hu. \end{cases}$$

This system can be written in the following vector form:

$$\boldsymbol{U}_t + \underbrace{\widetilde{\boldsymbol{F}}(\boldsymbol{U})_x + \widetilde{\boldsymbol{G}}(\boldsymbol{U})_y}_{\text{non-stiff terms}} + \underbrace{\widehat{\boldsymbol{F}}(\boldsymbol{U})_x + \widehat{\boldsymbol{G}}(\boldsymbol{U})_y}_{\text{stiff terms}} = \underbrace{\boldsymbol{S}(\boldsymbol{U})}_{\text{source terms}}$$

How to choose parameters α and $a(t)$?

Hyperbolic Flux Splitting

$$\underbrace{\boldsymbol{U}_t + \underbrace{\widetilde{\boldsymbol{F}}(\boldsymbol{U})_x + \widetilde{\boldsymbol{G}}(\boldsymbol{U})_y}_{\text{non-stiff terms}}}_{\text{nonlinear part}} + \underbrace{\underbrace{\widehat{\boldsymbol{F}}(\boldsymbol{U})_x + \widehat{\boldsymbol{G}}(\boldsymbol{U})_y}_{\text{stiff terms}} = \underbrace{\boldsymbol{S}(\boldsymbol{U})}_{\text{source terms}}}_{\text{linear part}}$$

Need to ensure: $\boldsymbol{U}_t + \widetilde{\boldsymbol{F}}(\boldsymbol{U})_x + \widetilde{\boldsymbol{G}}(\boldsymbol{U})_y = \boldsymbol{0}$ is both nonstiff and hyperbolic

Eigenvalues of the Jacobians $\partial\widetilde{\boldsymbol{F}}/\partial\boldsymbol{U}$ and $\partial\widetilde{\boldsymbol{G}}/\partial\boldsymbol{U}$:

$$\left\{u \pm \sqrt{(1-\alpha)u^2 + \alpha\frac{h-a(t)}{\varepsilon^2}},\ u\right\}, \quad \left\{v \pm \sqrt{(1-\alpha)v^2 + \alpha\frac{h-a(t)}{\varepsilon^2}},\ v\right\}$$

We then take

$$\alpha = \varepsilon^2 \quad \text{and} \quad a(t) = \min_{(x,y)\in\Omega} h(x,y,t)$$

Discretization of the Split System

$$\boldsymbol{U}^{n+1} = \boldsymbol{U}^n + \Delta t \underbrace{\widetilde{\boldsymbol{F}}(\boldsymbol{U})_x^n + \Delta t \widetilde{\boldsymbol{G}}(\boldsymbol{U})_y^n}_{\text{nonlinear part, explicit}}$$

$$+ \underbrace{\widehat{\boldsymbol{F}}(\boldsymbol{U})_x^{n+1} + \widehat{\boldsymbol{G}}(\boldsymbol{U})_y^{n+1} = \boldsymbol{S}(\boldsymbol{U})^{n+1}}_{\text{linear part, implicit}}$$

- Nonstiff nonlinear part is treated using the second-order central-upwind scheme
- Stiff linear part reduces to a linear elliptic equation for h^{n+1} and straigtforward computations of $(hu)^{n+1}$ and $(hv)^{n+1}$

$$\Delta t \le \nu \cdot \min\left(\frac{\Delta x}{\max\limits_{u,h}\left\{|u| + \sqrt{(1-\alpha)u^2 + \alpha\frac{h-a(t)}{\varepsilon^2}}\right\}}, \frac{\Delta y}{\max\limits_{v,h}\left\{|v| + \sqrt{(1-\alpha)v^2 + \alpha\frac{h-a(t)}{\varepsilon^2}}\right\}} \right)$$

Proof of the AP Property

Theorem. *The proposed hyperbolic flux splitting method coupled with the described fully discrete scheme is asymptotic preserving in the sense that it provides a consistent and stable discretization of the limiting system as the Froude number* $\varepsilon \to 0$.

Remark. In practice, the fully discrete scheme is both second-order accurate in space and time as we increase a temporal order of accuracy to the second one by implementing a two-stage globally stiffly accurate IMEX Runge-Kutta scheme ARS(2,2,2). The proof holds as well.

Example — 2-D Stationary Vortex

[E. Audusse, R. Klein, D. D. Nguyen, and S. Vater, 2011]

$$h(r,0) = 1+\varepsilon^2 \begin{cases} \frac{5}{2}(1+5\varepsilon^2)r^2 \\ \frac{1}{10}(1+5\varepsilon^2) + 2r - \frac{1}{2} - \frac{5}{2}r^2 + \varepsilon^2(4\ln(5r) + \frac{7}{2} - 20r + \frac{25}{2}r^2) \\ \frac{1}{5}(1 - 10\varepsilon + 4\varepsilon^2 \ln 2), \end{cases}$$

$$u(x,y,0) = -\varepsilon y \Upsilon(r), \quad v(x,y,0) = \varepsilon x \Upsilon(r), \quad \Upsilon(r) := \begin{cases} 5, & r < \frac{1}{5} \\ \frac{2}{r} - 5, & \frac{1}{5} \le r < \frac{2}{5} \\ 0, & r \ge \frac{2}{5}, \end{cases}$$

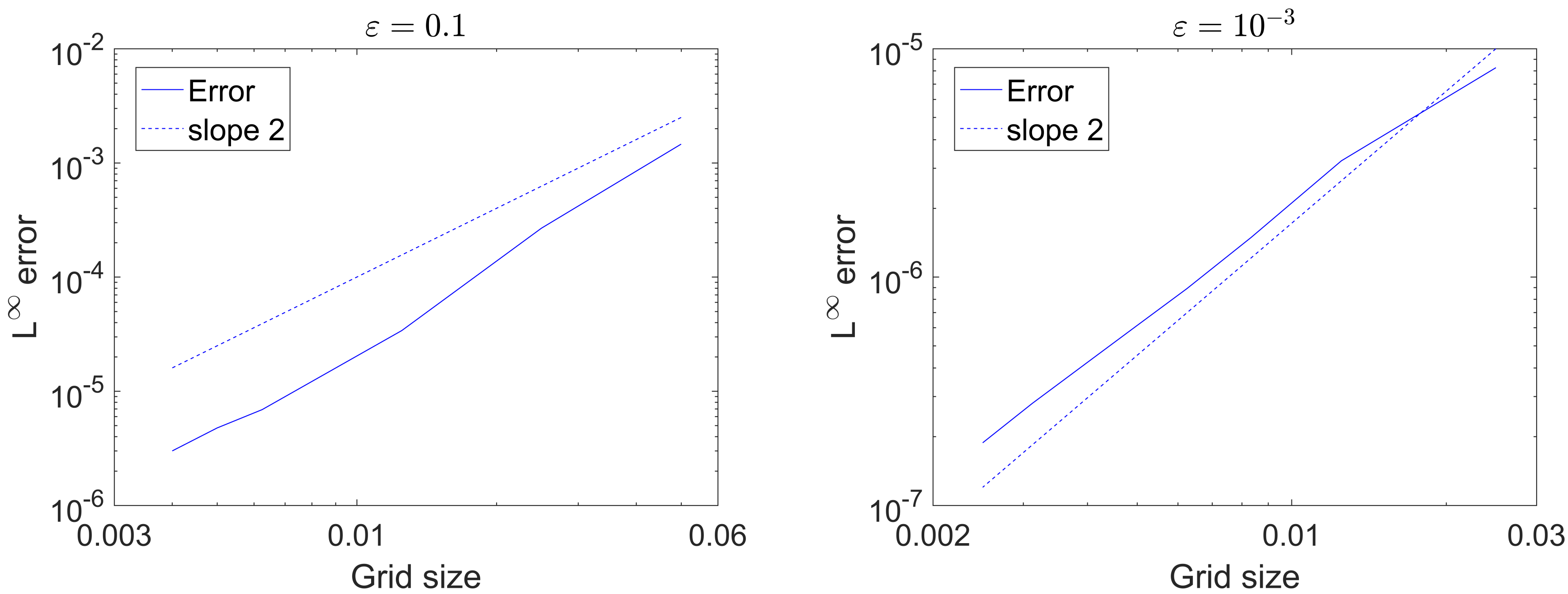
Domain: $[-1,1] \times [-1,1], \quad r := \sqrt{x^2+y^2}$

Boundary conditions: a zero-order extrapolation in both x- and y-directions

Numerical Tests:

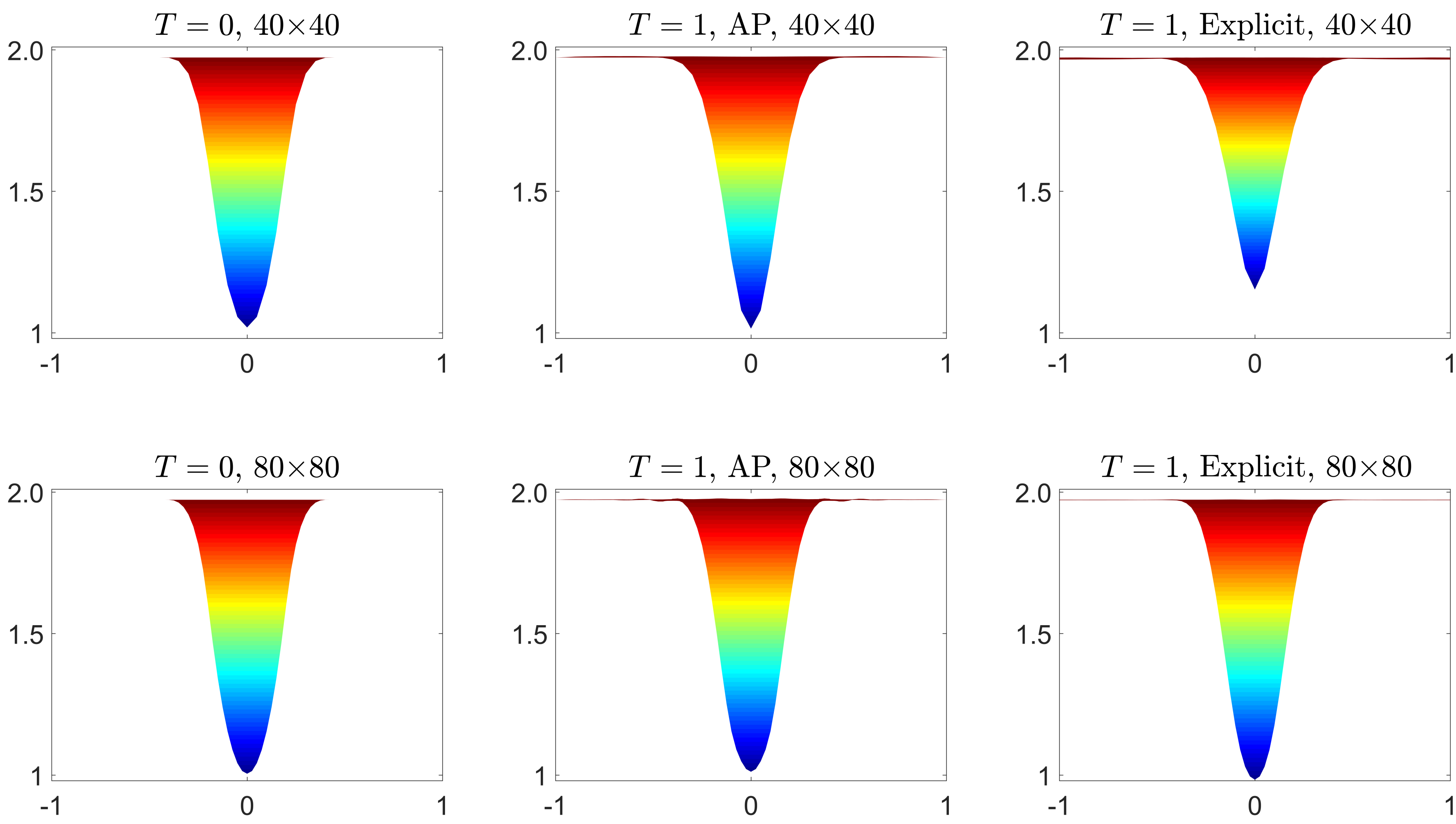
- Experimental order of convergence
- Comparison of non-AP and AP methods for various values of ε

Experimental order of convergence

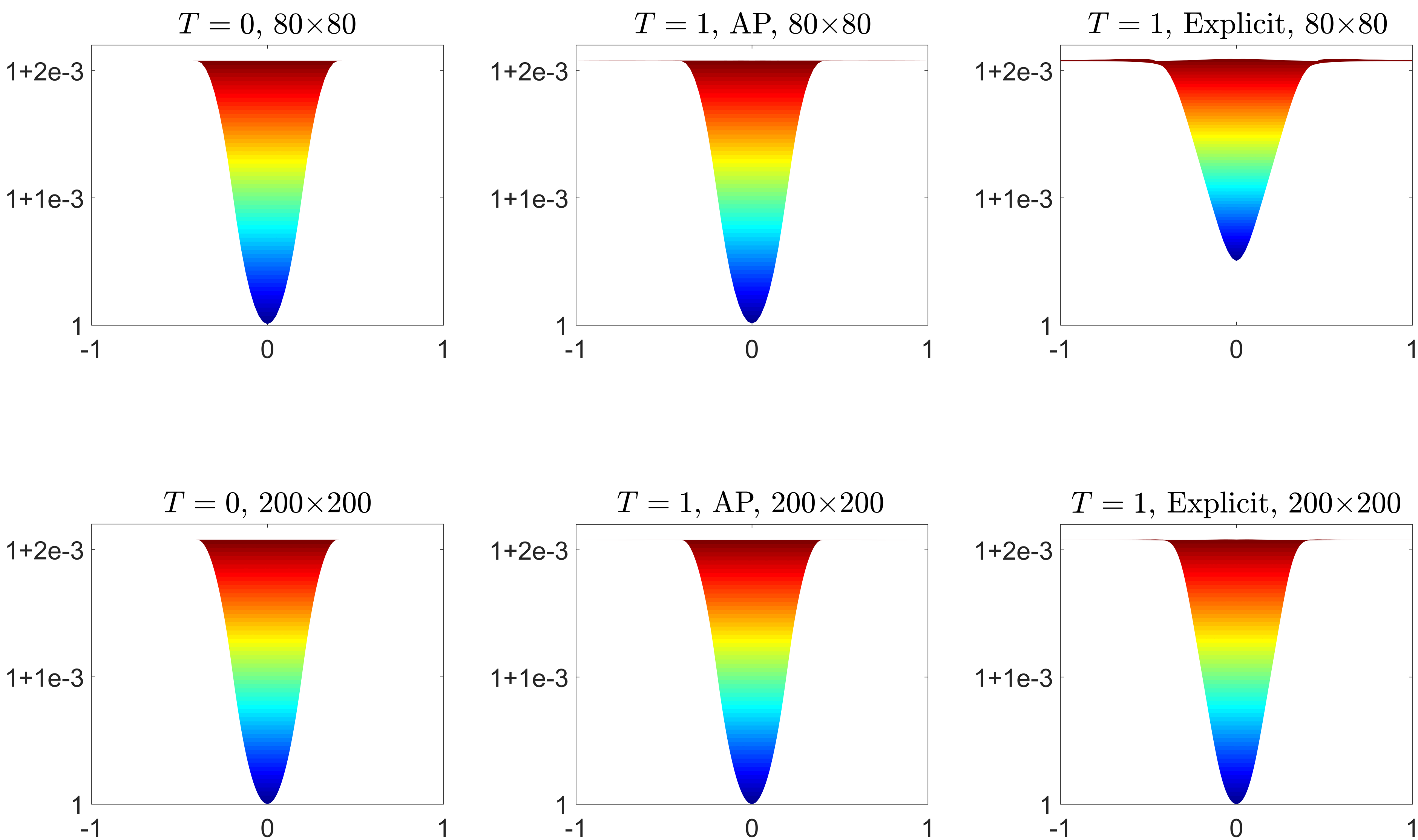


L^∞-errors for h computed using the AP scheme on several different grids for $\varepsilon = 0.1$ (left) and 10^{-3}

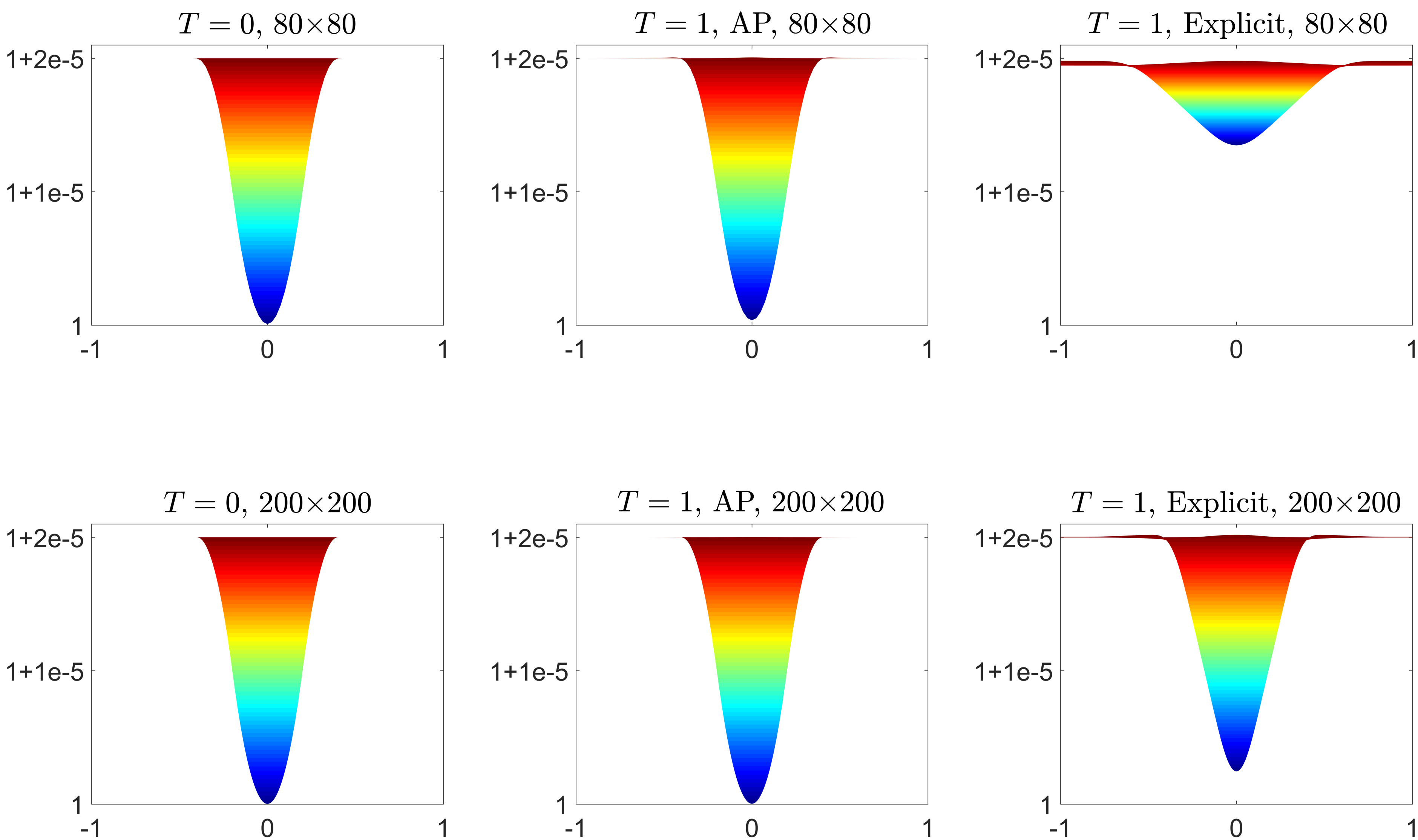
Comparison of non-AP and AP methods, $\varepsilon = 1$



Comparison of non-AP and AP methods, $\varepsilon = 0.1$



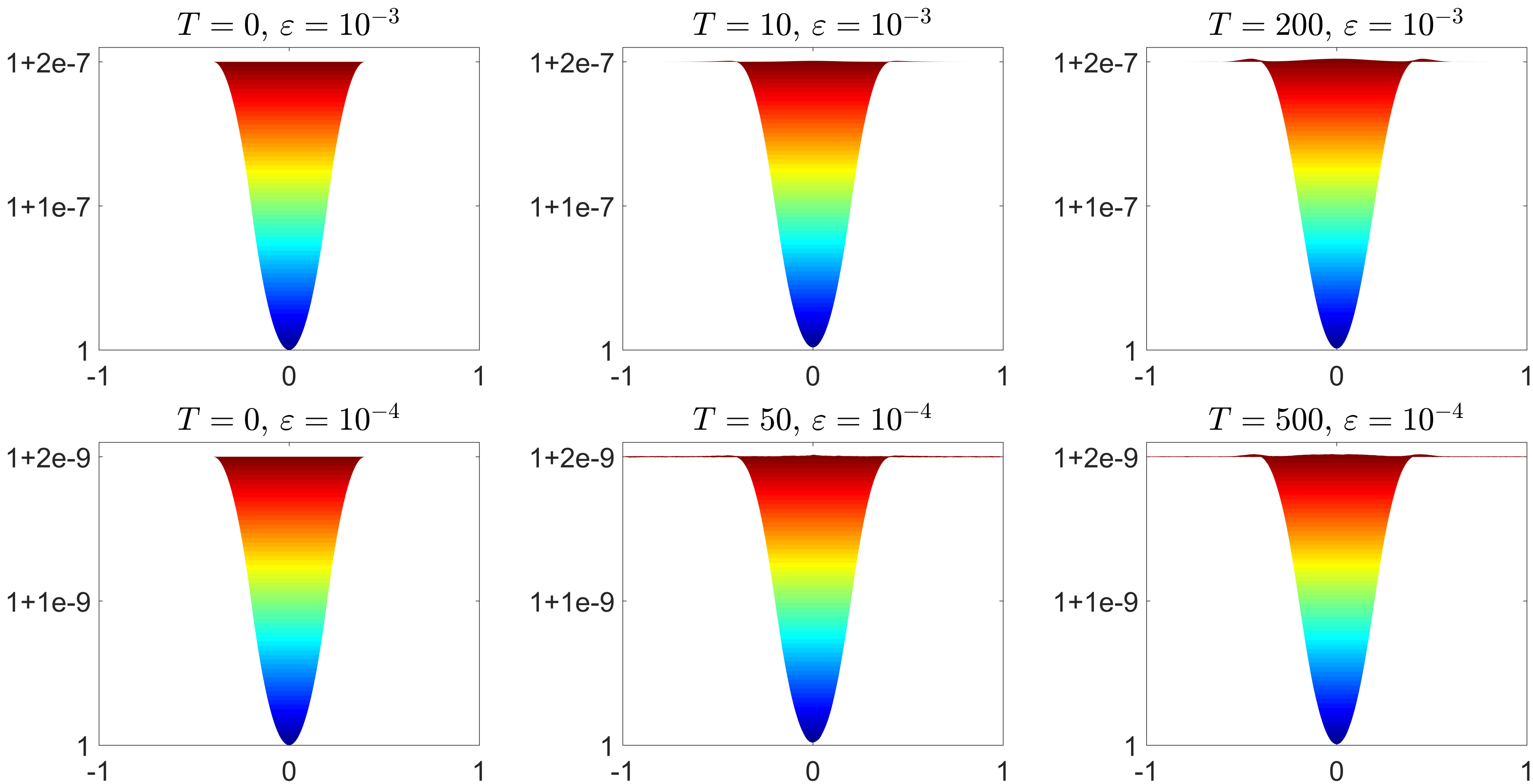
Comparison of non-AP and AP methods, $\varepsilon = 0.01$



Comparison of non-AP and AP methods, CPU times

	$\varepsilon = 1$		$\varepsilon = 0.1$		$\varepsilon = 0.01$	
Grid	AP	Explicit	AP	Explicit	AP	Explicit
40×40	0.18 s	0.16 s	0.06 s	1.25 s	0.03 s	10.53 s
80×80	1.57 s	1.32 s	0.29 s	4.73 s	0.18 s	47.0 s
200×200	24.11 s	21.36 s	5.36 s	163.36 s	3.37 s	804.15 s

Smaller values: $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$



Smaller times: 200×200, larger times: 500×500

THANK YOU!