Central-Upwind Schemes for Shallow Water Models

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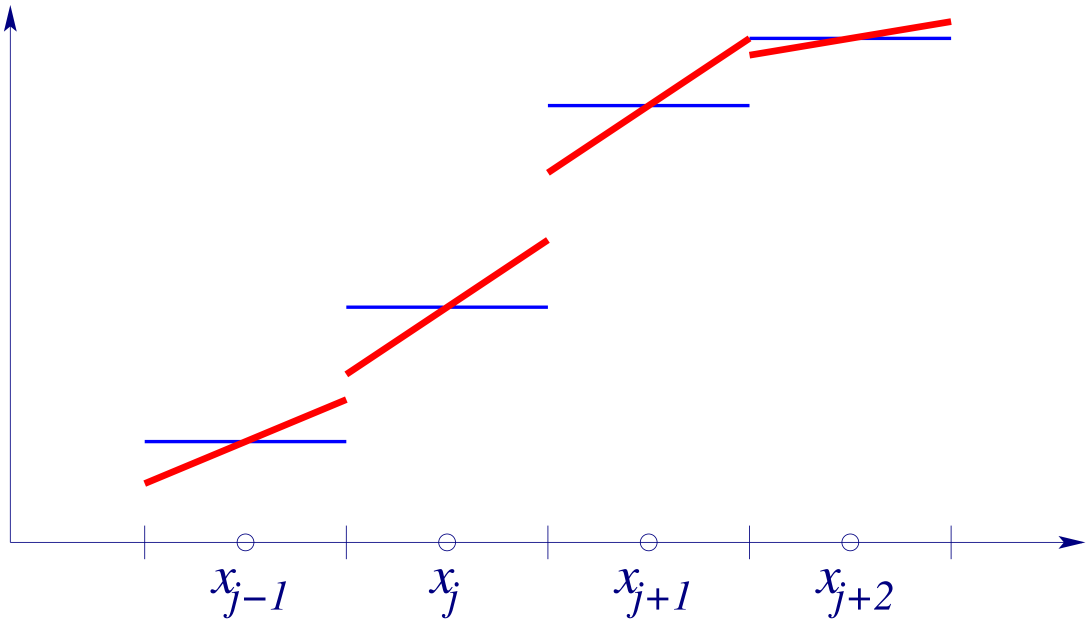
Finite-Volume Methods

1-D System: $\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U})_x = \boldsymbol{0}$

$\overline{\boldsymbol{U}}_j(t) \approx \dfrac{1}{\Delta x} \displaystyle\int_{C_j} \boldsymbol{U}(x,t)\,\mathrm{d}x$: cell averages over $C_j := (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$

This solution is approximated by a piecewise linear (conservative, second-order accurate, non-oscillatory) reconstruction:

$\widetilde{\boldsymbol{U}}(x) = \overline{\boldsymbol{U}}_j + (\boldsymbol{U}_x)_j(x - x_j)$ for $x \in C_j$



For example,

$$\boxed{(\boldsymbol{U}_x)_j = \text{minmod}\left(\theta\frac{\overline{\boldsymbol{U}}_j - \overline{\boldsymbol{U}}_{j-1}}{\Delta x}, \frac{\overline{\boldsymbol{U}}_{j+1} - \overline{\boldsymbol{U}}_{j-1}}{2\Delta x}, \theta\frac{\overline{\boldsymbol{U}}_{j+1} - \overline{\boldsymbol{U}}_j}{\Delta x}\right)} \quad \theta \in [1, 2]$$

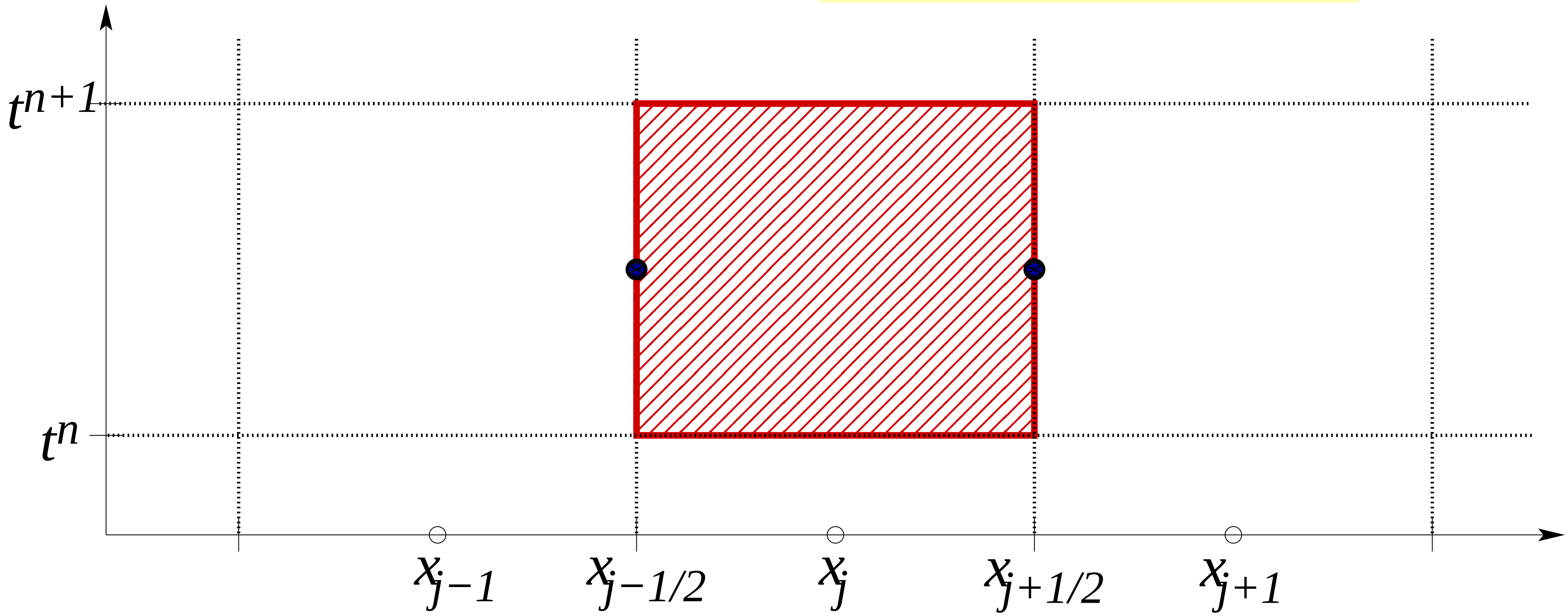
where the minmod function is defined as:

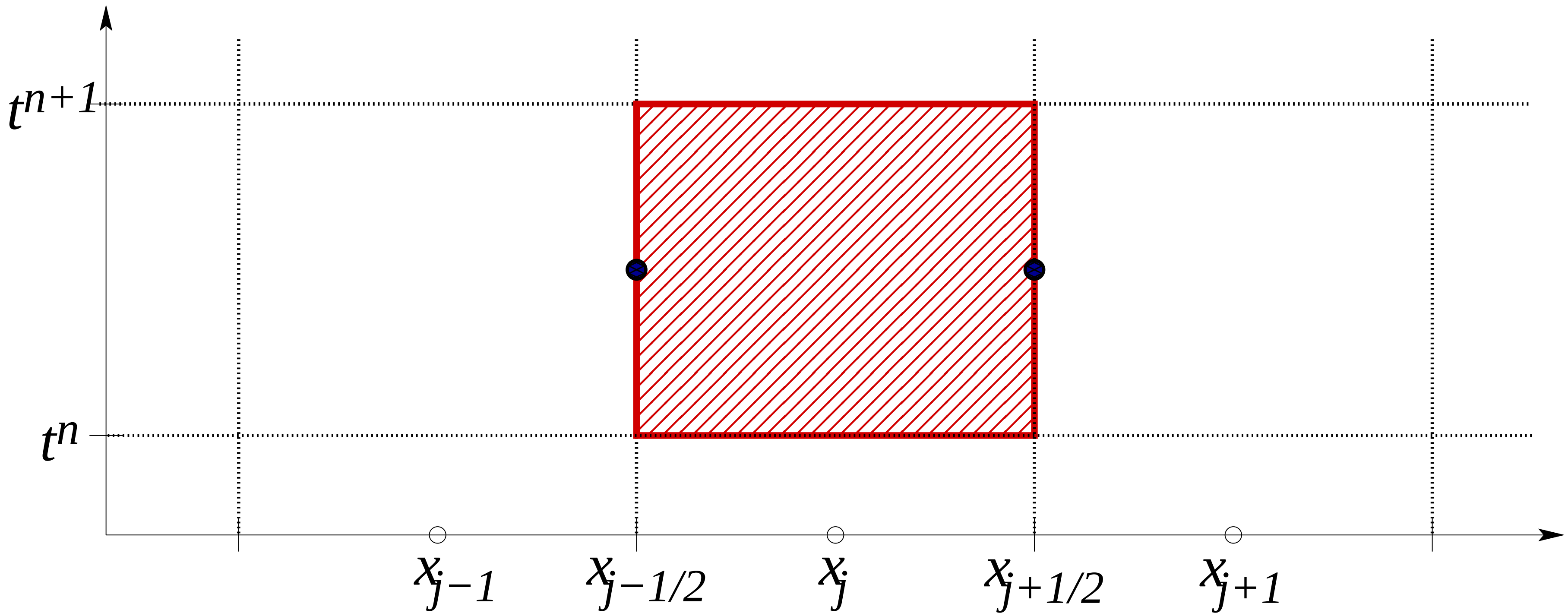
$$\boxed{\text{minmod}(z_1, z_2, ...) := \begin{cases} \min_j\{z_j\}, & \text{if} \quad z_j > 0 \quad \forall j \\ \max_j\{z_j\}, & \text{if} \quad z_j < 0 \quad \forall j \\ 0, & \text{otherwise} \end{cases}}$$

Godunov-type upwind schemes are designed by integrating

$$\boldsymbol{U}_t + \boldsymbol{f}(\boldsymbol{U})_x = \boldsymbol{0}$$

over the space-time control volumes $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \times [t^n, t^{n+1}]$

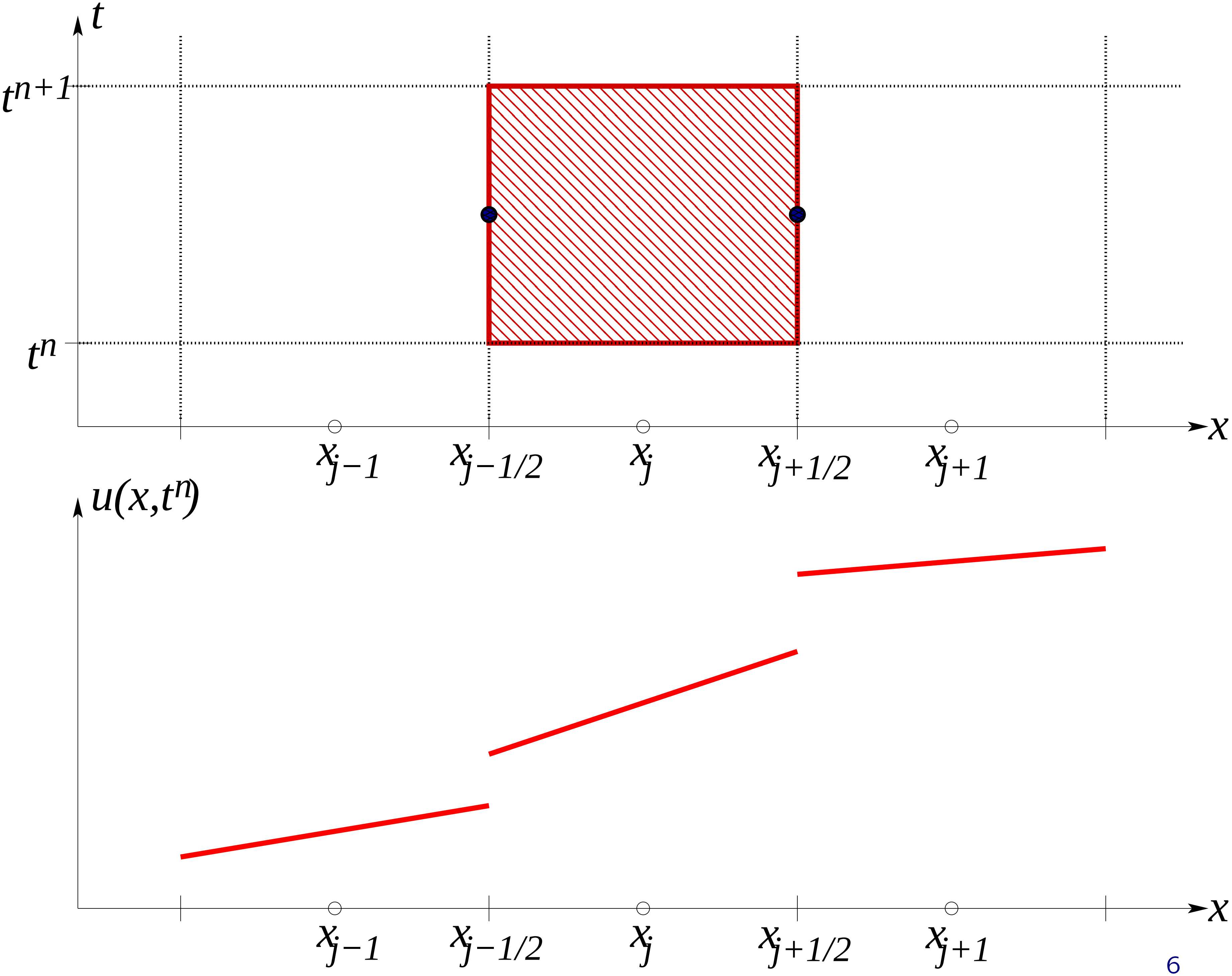




$$\overline{\boldsymbol{U}}_j^{n+1} = \overline{\boldsymbol{U}}_j^{n} - \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} \left[\boldsymbol{f}(\boldsymbol{U}(x_{j+\frac{1}{2}}, t)) - \boldsymbol{f}(\boldsymbol{U}(x_{j-\frac{1}{2}}, t)) \right] \mathrm{d}t$$

In order to evaluate the flux integrals on the RHS, one has to (approximately) solve the generalized Riemann problem.

This may be hard or even impossible...



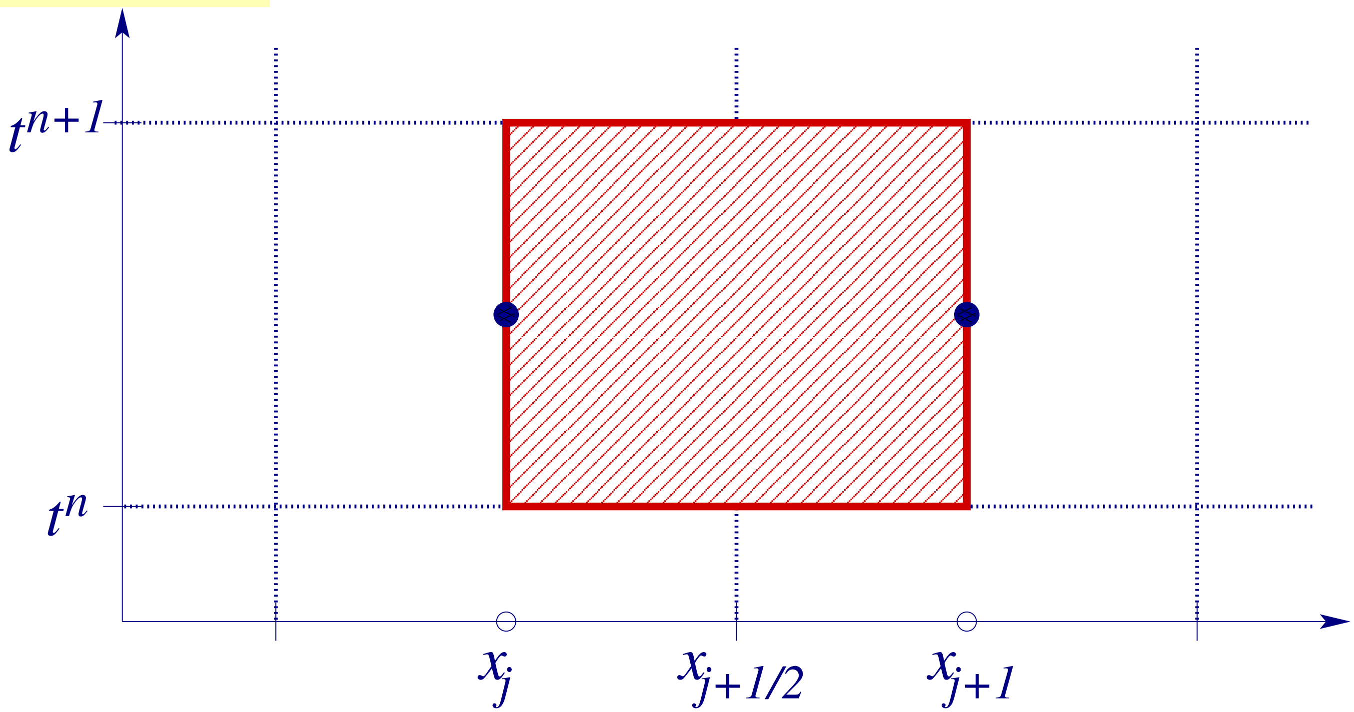
t
t^{n+1}
t^n
x
x_{j-1}
x_{j-1/2}
x_j
x_{j+1/2}
x_{j+1}
u(x,t^n)
x
x_{j-1}
x_{j-1/2}
x_j
x_{j+1/2}
x_{j+1}

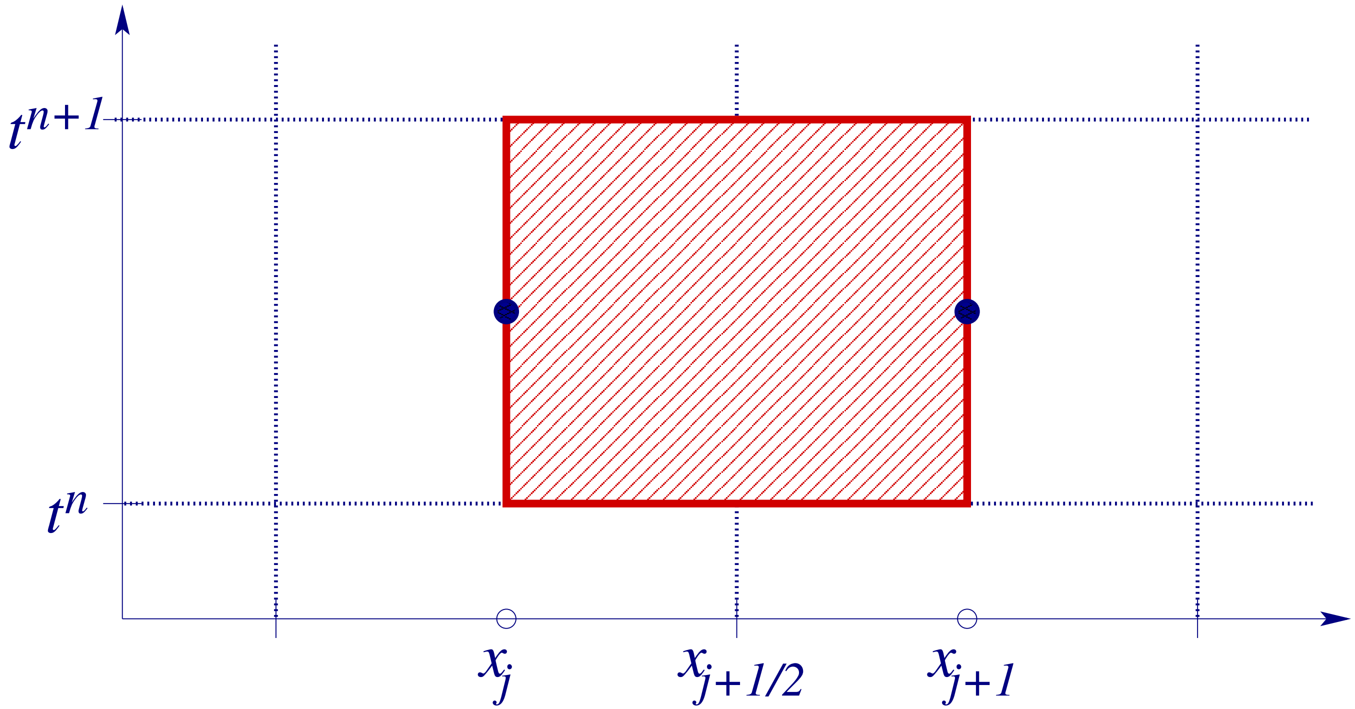
Nessyahu-Tadmor Scheme

The Nessyahu-Tadmor [Nessyahu, Tadmor; 1990] scheme is a central Godunov-type scheme. It is designed by integrating

$$\boldsymbol{U}_t + \boldsymbol{f}(\boldsymbol{U})_x = \mathbf{0}$$

over the different set of staggered space-time control volumes $[x_j, x_{j+1}] \times [t^n, t^{n+1}]$ containing the Riemann fans





$$\overline{\boldsymbol{U}}^{n+1}_{j+\frac{1}{2}} = \frac{1}{\Delta x}\int\limits_{x_j}^{x_{j+1}} \widetilde{\boldsymbol{U}}^n(x)\,\mathrm{d}x - \frac{1}{\Delta x}\int\limits_{t^n}^{t^{n+1}} \Big[\boldsymbol{f}(\boldsymbol{U}(x_{j+1},t)) - \boldsymbol{f}(\boldsymbol{U}(x_j,t))\Big]\mathrm{d}t$$

Due to the finite speed of propagation, this can be reduced to:

$$\overline{\boldsymbol{U}}^{n+1}_{j+\frac{1}{2}} = \frac{\overline{\boldsymbol{U}}^n_j + \overline{\boldsymbol{U}}^n_{j+1}}{2} + \frac{\Delta x}{8}\Big((\boldsymbol{U}_x)^n_j - (\boldsymbol{U}_x)^n_{j+1}\Big) - \frac{\Delta t}{\Delta x}\Big[\boldsymbol{f}(\boldsymbol{U}^{n+\frac{1}{2}}_{j+1}) - \boldsymbol{f}(\boldsymbol{U}^{n+\frac{1}{2}}_{j})\Big]$$

Values of $\boldsymbol{U}$ at $t = t^{n+\frac{1}{2}}$ are approximated using the Taylor expansion:

$$\boldsymbol{U}_j^{n+\frac{1}{2}} \approx \widetilde{\boldsymbol{U}}^n(x_j) + \frac{\Delta t}{2}\boldsymbol{U}_t(x_j, t^n)$$

- $\widetilde{\boldsymbol{U}}^n(x) = \overline{\boldsymbol{U}}_j^n + (\boldsymbol{U}_x)_j^n(x - x_j) \Longrightarrow \boxed{\widetilde{\boldsymbol{U}}^n(x_j) = \overline{\boldsymbol{U}}_j^n}$

- $\boxed{\boldsymbol{U}_t(x_j, t^n) = -\boldsymbol{f}(\overline{\boldsymbol{U}}_j^n)_x}$

The space derivatives $\boldsymbol{f}_x$ are computed using the (minmod) limiter:

$$\boldsymbol{f}(\overline{\boldsymbol{U}}_j^n)_x = \text{minmod}\left(\theta\,\frac{\boldsymbol{f}(\overline{\boldsymbol{U}}_j^n) - \boldsymbol{f}(\overline{\boldsymbol{U}}_{j-1}^n)}{\Delta x}, \frac{\boldsymbol{f}(\overline{\boldsymbol{U}}_{j+1}^n) - \boldsymbol{f}(\overline{\boldsymbol{U}}_{j-1}^n)}{2\Delta x}, \theta\,\frac{\boldsymbol{f}(\overline{\boldsymbol{U}}_{j+1}^n) - \boldsymbol{f}(\overline{\boldsymbol{U}}_j^n)}{\Delta x}\right)$$

Higher-Order and Multidimensional Staggered Central Schemes

[Arminjon, Viallon, Madrane; 1997]

[Jiang, Tadmor; 1998]

[Liu, Tadmor; 1998]

[Bianco, Puppo, Russo; 1999]

[Levy, Puppo, Russo; 1999, 2000, 2002]

[Lie, Noelle; 2000]

Central-Upwind Schemes

Goal: **to reduce numerical dissipation of central schemes**

Example — Numerical Dissipation of the Staggered LxF Scheme

$$u^{n+1}_{j+\frac{1}{2}} = \frac{u^n_{j+1} + u^n_j}{2} - \frac{\Delta t}{\Delta x}\Big[f(u^n_{j+1}) - f(u^n_j)\Big]$$

$$u^{n+1}_{j+\frac{1}{2}} - u^n_{j+\frac{1}{2}} + \frac{\Delta t}{\Delta x}\Big[f(u^n_{j+1}) - f(u^n_j)\Big] = \frac{u^n_{j+1} - 2u^n_{j+\frac{1}{2}} + u^n_j}{2}$$

$$\frac{u^{n+1}_{j+\frac{1}{2}} - u^n_{j+\frac{1}{2}}}{\Delta t} + \frac{f(u^n_{j+1}) - f(u^n_j)}{\Delta x} = \boxed{\frac{(\Delta x)^2}{8\Delta t}} \cdot \frac{u^n_{j+1} - 2u^n_{j+\frac{1}{2}} + u^n_j}{(\Delta x/2)^2}$$

- As Δt decreases, the numerical dissipation increases
- As $\Delta t \sim (\Delta x)^2$, the LxF scheme is inconsistent
- As $\Delta t \to 0$, the numerical dissipation blows up

Central-Upwind Schemes

Godunov-type central schemes with a built-in upwind nature

[Kurganov, Tadmor; 2000]

[Kurganov, Petrova; 2001]

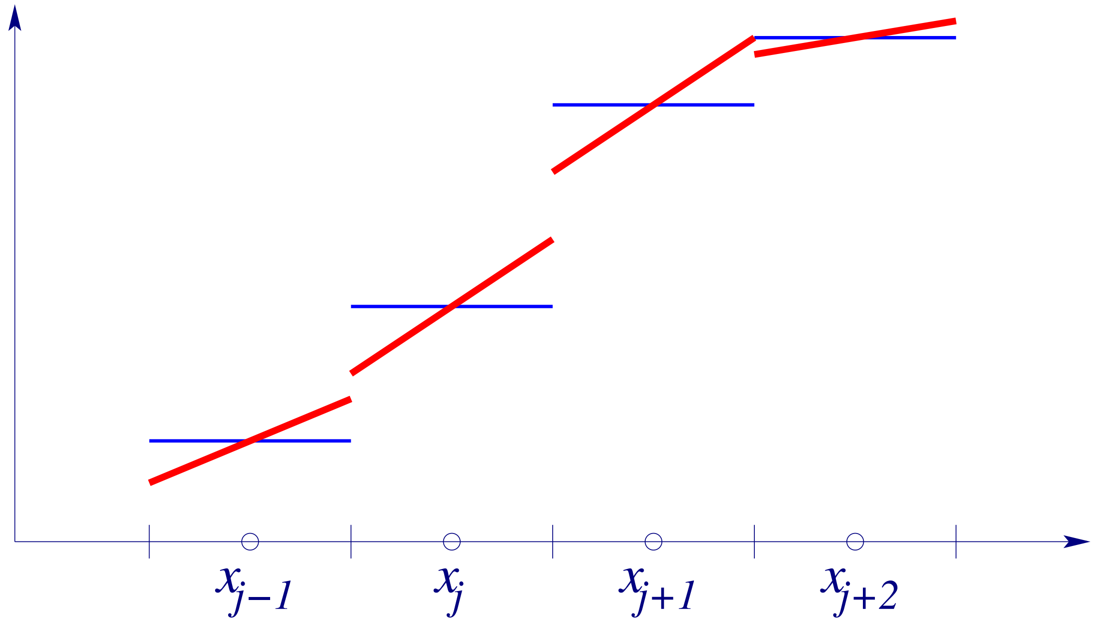
[Kurganov, Noelle, Petrova; 2001]

[Kurganov, Tadmor; 2002]

[Kurganov, Petrova; 2005]

[Kurganov, Lin; 2007]

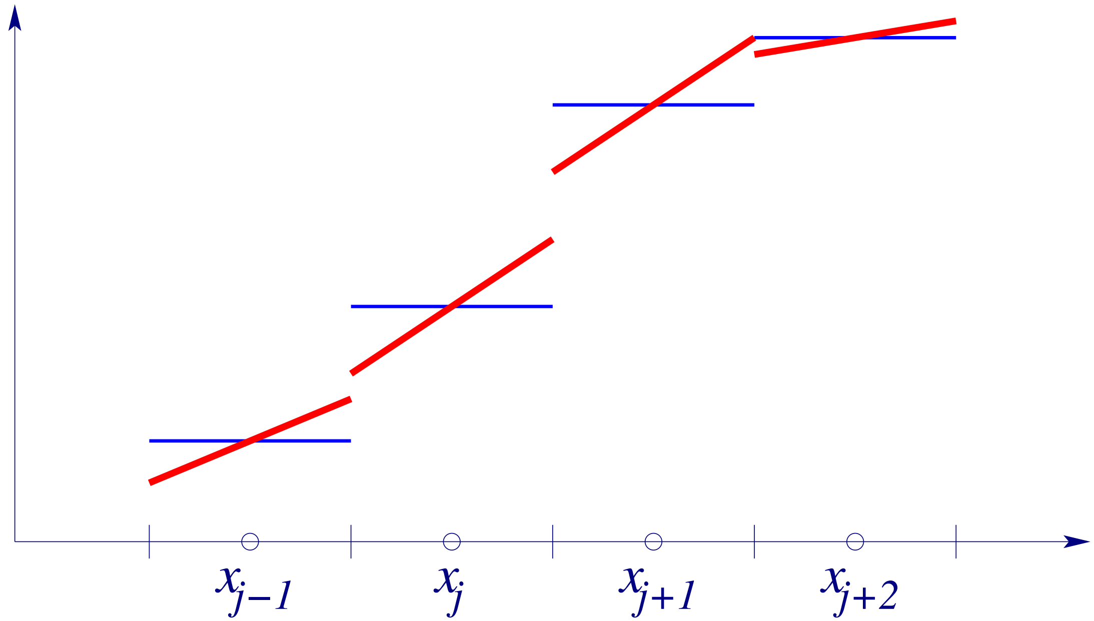
[Kurganov, Prugger, Wu; 2017]



$$\widetilde{\boldsymbol{U}}^{\,n}(x) = \overline{\boldsymbol{U}}_j^{\,n} + (\boldsymbol{U}_x)_j^n (x - x_j) \quad \text{for } x \in C_j$$

$$\boldsymbol{U}^-_{j+\frac{1}{2}} := \lim_{x \to x_{j+\frac{1}{2}}-} \widetilde{\boldsymbol{U}}(x, t^n) = \overline{\boldsymbol{U}}_j^n + \frac{\Delta x}{2} (\boldsymbol{U}_x)_j^n$$

$$\boldsymbol{U}^+_{j+\frac{1}{2}} := \lim_{x \to x_{j+\frac{1}{2}}+} \widetilde{\boldsymbol{U}}(x, t^n) = \overline{\boldsymbol{U}}_{j+1}^n - \frac{\Delta x}{2} (\boldsymbol{U}_x)_{j+1}^n$$



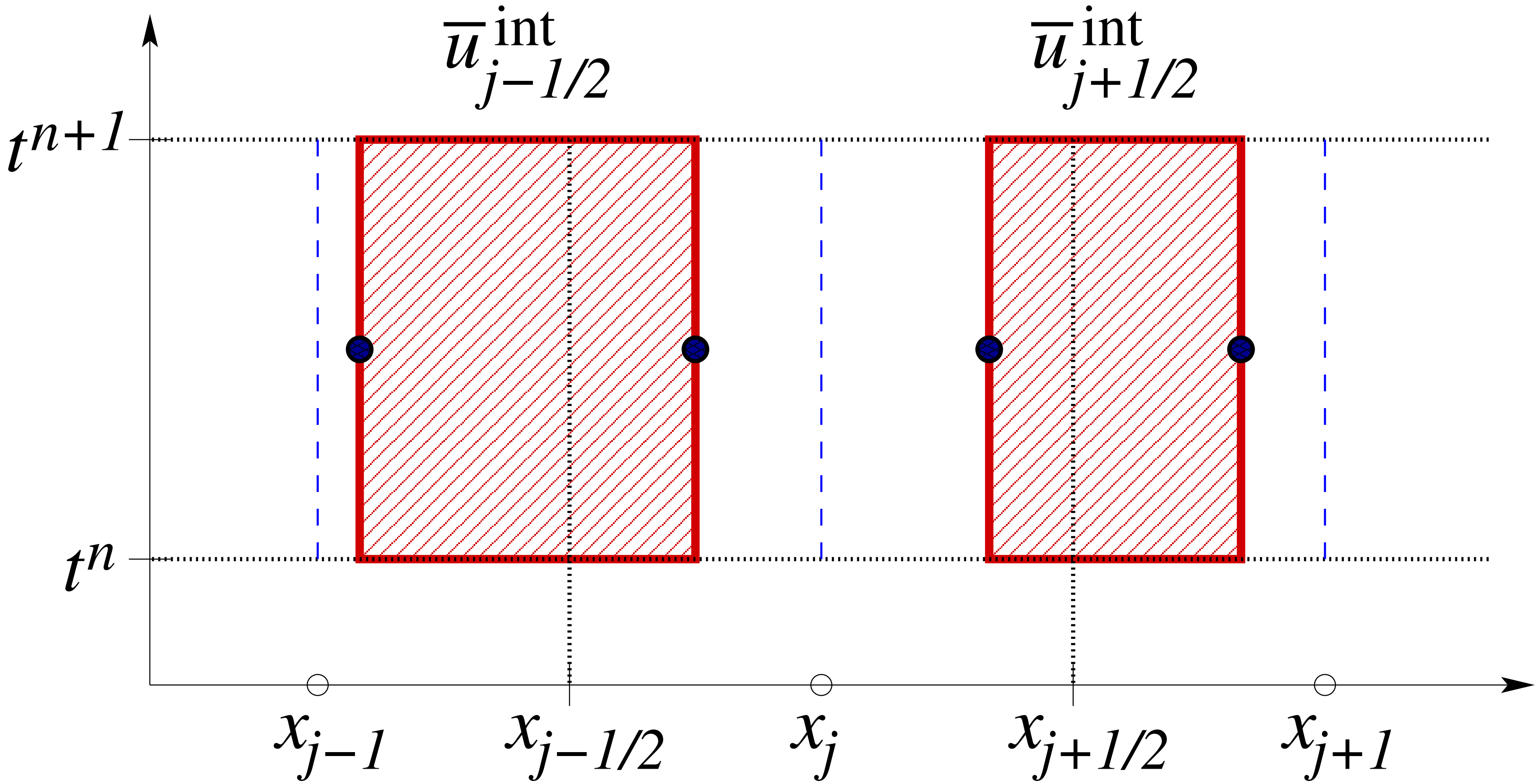
The discontinuities appearing at the reconstruction step at the interface points $\{x_{j+\frac{1}{2}}\}$ propagate at finite speeds estimated by:

$$a^{+}_{j+\frac{1}{2}} := \max\left\{\lambda_N\left(\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}(\boldsymbol{U}^{-}_{j+\frac{1}{2}})\right), \lambda_N\left(\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}(\boldsymbol{U}^{+}_{j+\frac{1}{2}})\right), 0\right\}$$

$$a^{-}_{j+\frac{1}{2}} := \min\left\{\lambda_1\left(\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}(\boldsymbol{U}^{-}_{j+\frac{1}{2}})\right), \lambda_1\left(\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}(\boldsymbol{U}^{+}_{j+\frac{1}{2}})\right), 0\right\}$$

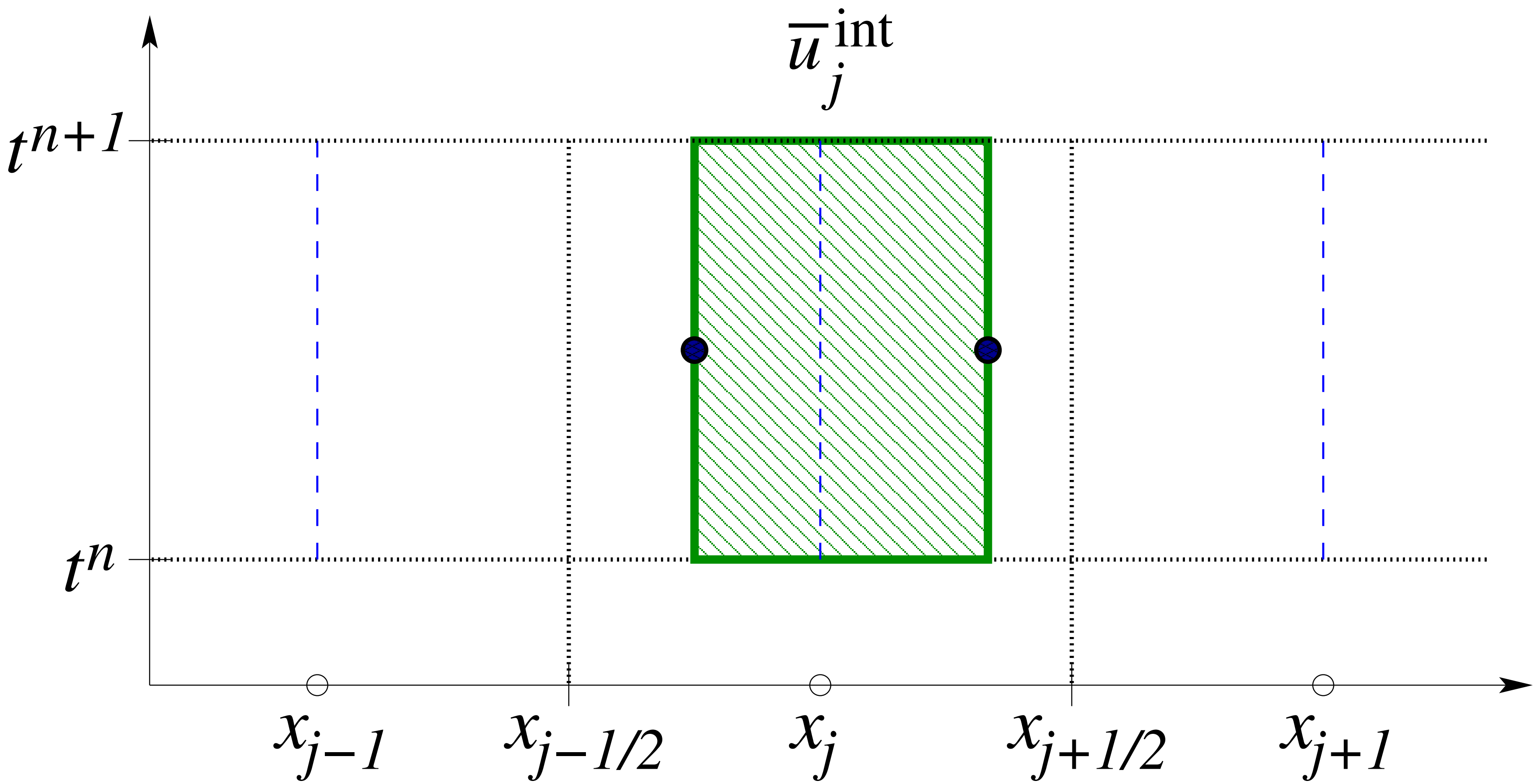
$\lambda_1 < \lambda_2 < \ldots < \lambda_N$: N eigenvalues of the Jacobian $\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}$

Idea: **Select control volumes according to the size of each Riemann fan**



$$\left[x_{j-\frac{1}{2}} + a^{-}_{j-\frac{1}{2}}\Delta t, x_{j-\frac{1}{2}} + a^{+}_{j-\frac{1}{2}}\Delta t\right] \times [t^n, t^{n+1}]$$

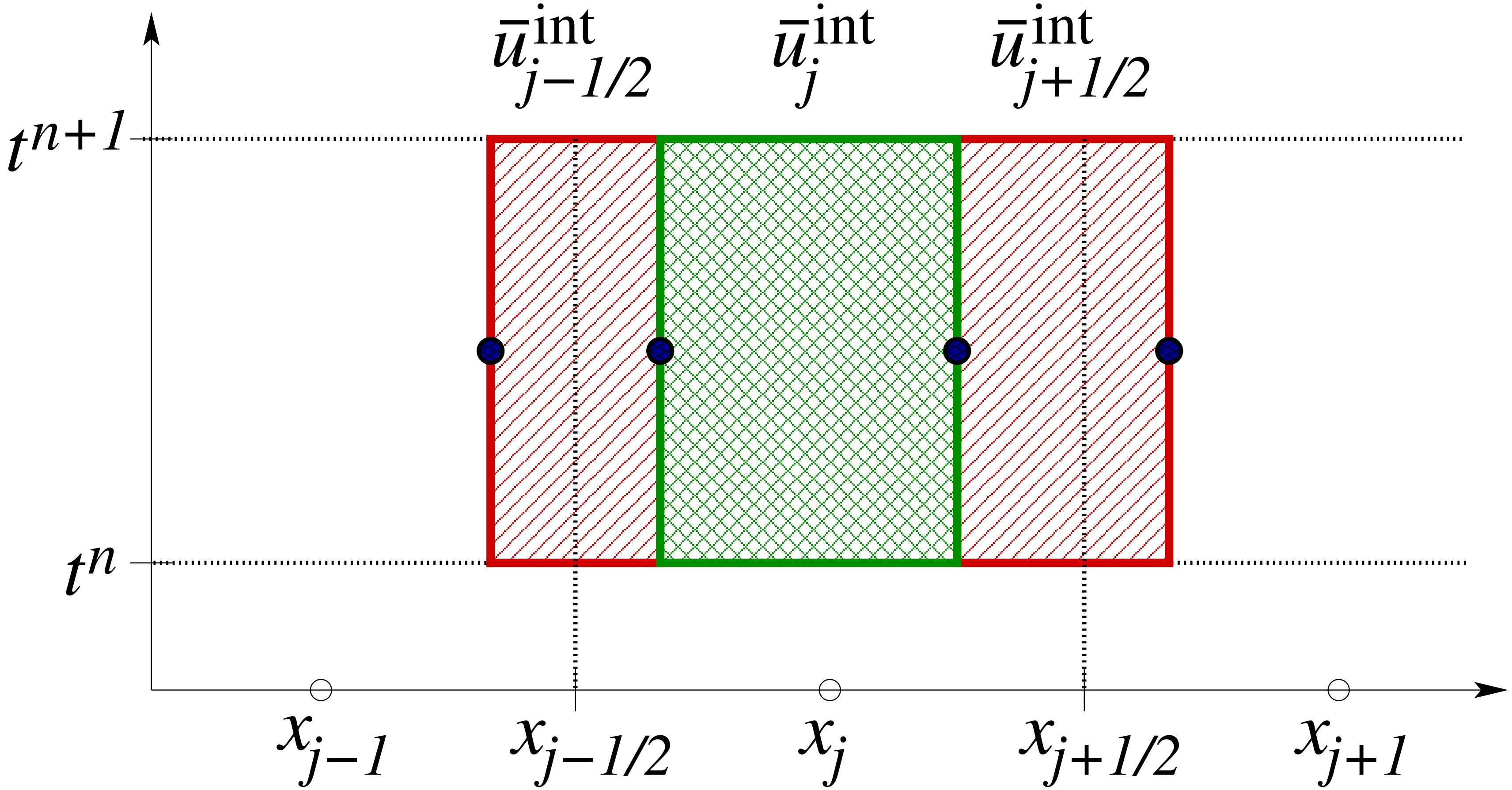
$$\left[x_{j+\frac{1}{2}} + a^{-}_{j+\frac{1}{2}}\Delta t, x_{j+\frac{1}{2}} + a^{+}_{j+\frac{1}{2}}\Delta t\right] \times [t^n, t^{n+1}]$$

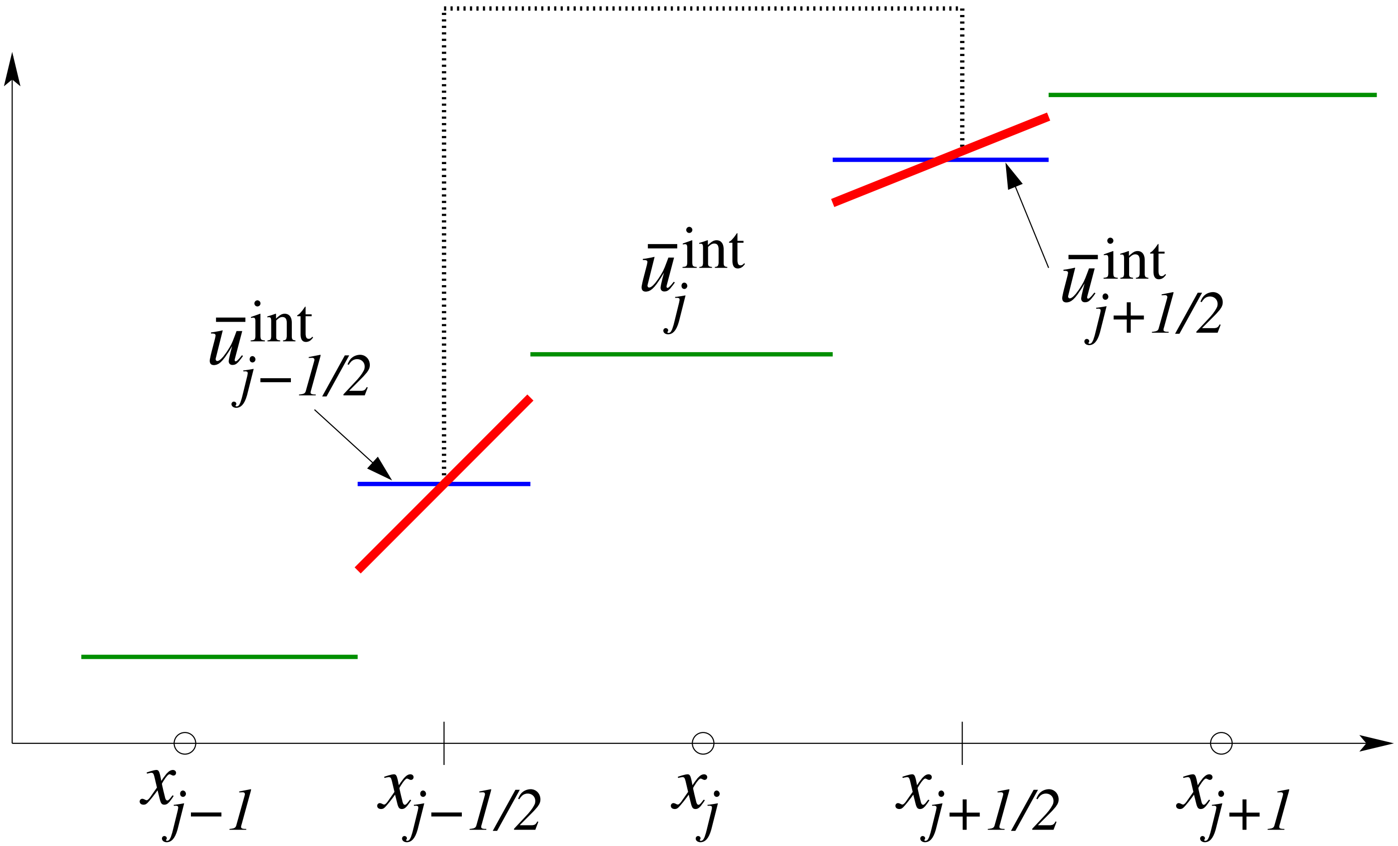


$$\left[x_{j-\frac{1}{2}} + a^{+}_{j-\frac{1}{2}}\Delta t, x_{j+\frac{1}{2}} + a^{-}_{j-\frac{1}{2}}\Delta t\right] \times [t^n, t^{n+1}]$$

Final Step: **Projection onto the Original Grid**

A piecewise linear interpolant, $\widetilde{\boldsymbol{U}}^{\,\mathrm{int}}(x)$, reconstructed from the evolved intermediate cell averages $\{\overline{\boldsymbol{U}}_j^{\,\mathrm{int}}\}$ and $\{\overline{\boldsymbol{U}}_{j+\frac{1}{2}}^{\,\mathrm{int}}\}$, is projected back onto the original grid by averaging it over the intervals $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$.





New projected cell averages:

$$\overline{\boldsymbol{U}}_j^{\,n+1} = \frac{a^+_{j-\frac12}\Delta t}{\Delta x}\overline{\boldsymbol{U}}^{\text{int}}_{j-\frac12} + \left[1 + \frac{\left(a^-_{j-\frac12} - a^+_{j+\frac12}\right)\Delta t}{\Delta x}\right]\overline{\boldsymbol{U}}^{\text{int}}_j - \frac{a^-_{j+\frac12}\Delta t}{\Delta x}\overline{\boldsymbol{U}}^{\text{int}}_{j+\frac12}$$
$$+\frac{(\Delta t)^2}{2\Delta x}\left[\boxed{\boxed{(\boldsymbol{U}_x)^{\text{int}}_{j+\frac12}}}\, a^+_{j+\frac12}a^-_{j+\frac12} - \boxed{\boxed{(\boldsymbol{U}_x)^{\text{int}}_{j-\frac12}}}\, a^+_{j-\frac12}a^-_{j-\frac12}\right]$$

1-D Semi-Discrete Central-Upwind Scheme

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\overline{\boldsymbol{U}}_j(t^n) &= \lim_{\Delta t\to 0}\frac{\overline{\boldsymbol{U}}_j^{\,n+1}-\overline{\boldsymbol{U}}_j^{\,n}}{\Delta t} = \frac{a^+_{j-\frac12}}{\Delta x}\lim_{\Delta t\to 0}\overline{\boldsymbol{U}}^{\,\mathrm{int}}_{j-\frac12} - \frac{a^-_{j+\frac12}}{\Delta x}\lim_{\Delta t\to 0}\overline{\boldsymbol{U}}^{\,\mathrm{int}}_{j+\frac12} \\
&+\frac{a^-_{j-\frac12}-a^+_{j+\frac12}}{\Delta x}\lim_{\Delta t\to 0}\overline{\boldsymbol{U}}^{\,\mathrm{int}}_j + \lim_{\Delta t\to 0}\left\{\frac{\overline{\boldsymbol{U}}^{\,\mathrm{int}}_j-\overline{\boldsymbol{U}}^{\,n}_j}{\Delta t}\right\} \\
&+\frac{1}{2\Delta x}\lim_{\Delta t\to 0}\left[\Delta t\left((\boldsymbol{U}_x)^{\mathrm{int}}_{j+\frac12}a^+_{j+\frac12}a^-_{j+\frac12}-(\boldsymbol{U}_x)^{\mathrm{int}}_{j-\frac12}a^+_{j-\frac12}a^-_{j-\frac12}\right)\right]
\end{aligned}
$$

We then substitute $\boldsymbol{U}^{\mathrm{int}}_{j\pm\frac12}$, $\boldsymbol{U}^{\mathrm{int}}_j$ and $(\boldsymbol{U}_x)^{\mathrm{int}}_{j\pm\frac12}$ into here to obtain the 1-D semi-discrete central-upwind scheme

(for details see [Kurganov, Lin; 2007])

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\boldsymbol{U}}_j(t) = -\frac{\boldsymbol{H}_{j+\frac{1}{2}}(t) - \boldsymbol{H}_{j-\frac{1}{2}}(t)}{\Delta x}$$

The central-upwind numerical flux is:

$$\boldsymbol{H}_{j+\frac{1}{2}} = \frac{a^+_{j+\frac{1}{2}}\boldsymbol{F}(\boldsymbol{U}^-_{j+\frac{1}{2}}) - a^-_{j+\frac{1}{2}}\boldsymbol{F}(\boldsymbol{U}^+_{j+\frac{1}{2}})}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} + a^+_{j+\frac{1}{2}}a^-_{j+\frac{1}{2}}\left[\frac{\boldsymbol{U}^+_{j+\frac{1}{2}} - \boldsymbol{U}^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} - \boxed{\boldsymbol{d}_{j+\frac{1}{2}}}\right]$$

The built-in "anti-diffusion" term is:

$$\boldsymbol{d}_{j+\frac{1}{2}} = \frac{1}{2}\lim_{\Delta t\to 0}\left\{\Delta t(\boldsymbol{U}_x)^{\mathrm{int}}_{j+\frac{1}{2}}\right\} = \mathrm{minmod}\left(\frac{\boldsymbol{U}^+_{j+\frac{1}{2}} - \boldsymbol{U}^*_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}, \frac{\boldsymbol{U}^*_{j+\frac{1}{2}} - \boldsymbol{U}^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}\right)$$

The intermediate values $\boldsymbol{U}^*_{j+\frac{1}{2}}$ are:

$$\boldsymbol{U}^*_{j+\frac{1}{2}} = \lim_{\Delta t\to 0}\overline{\boldsymbol{U}}^{\mathrm{int}}_{j+\frac{1}{2}} = \frac{a^+_{j+\frac{1}{2}}\boldsymbol{U}^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}\boldsymbol{U}^-_{j+\frac{1}{2}} - \left\{\boldsymbol{F}(\boldsymbol{U}^+_{j+\frac{1}{2}}) - \boldsymbol{F}(\boldsymbol{U}^-_{j+\frac{1}{2}})\right\}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}$$

Remarks

1. $\boldsymbol{d}_{j+\frac{1}{2}} \equiv 0$ corresponds to the original central-upwind scheme from [Kurganov, Noelle, Petrova; 2001]

$\boldsymbol{d}_{j+\frac{1}{2}} \equiv 0$ and $a^+_{j+\frac{1}{2}} \equiv -a^-_{j+\frac{1}{2}}$ correspond to the scheme from [Kurganov, Tadmor; 2000]

2. For the system of balance laws

$$\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U})_x = \boldsymbol{S}$$

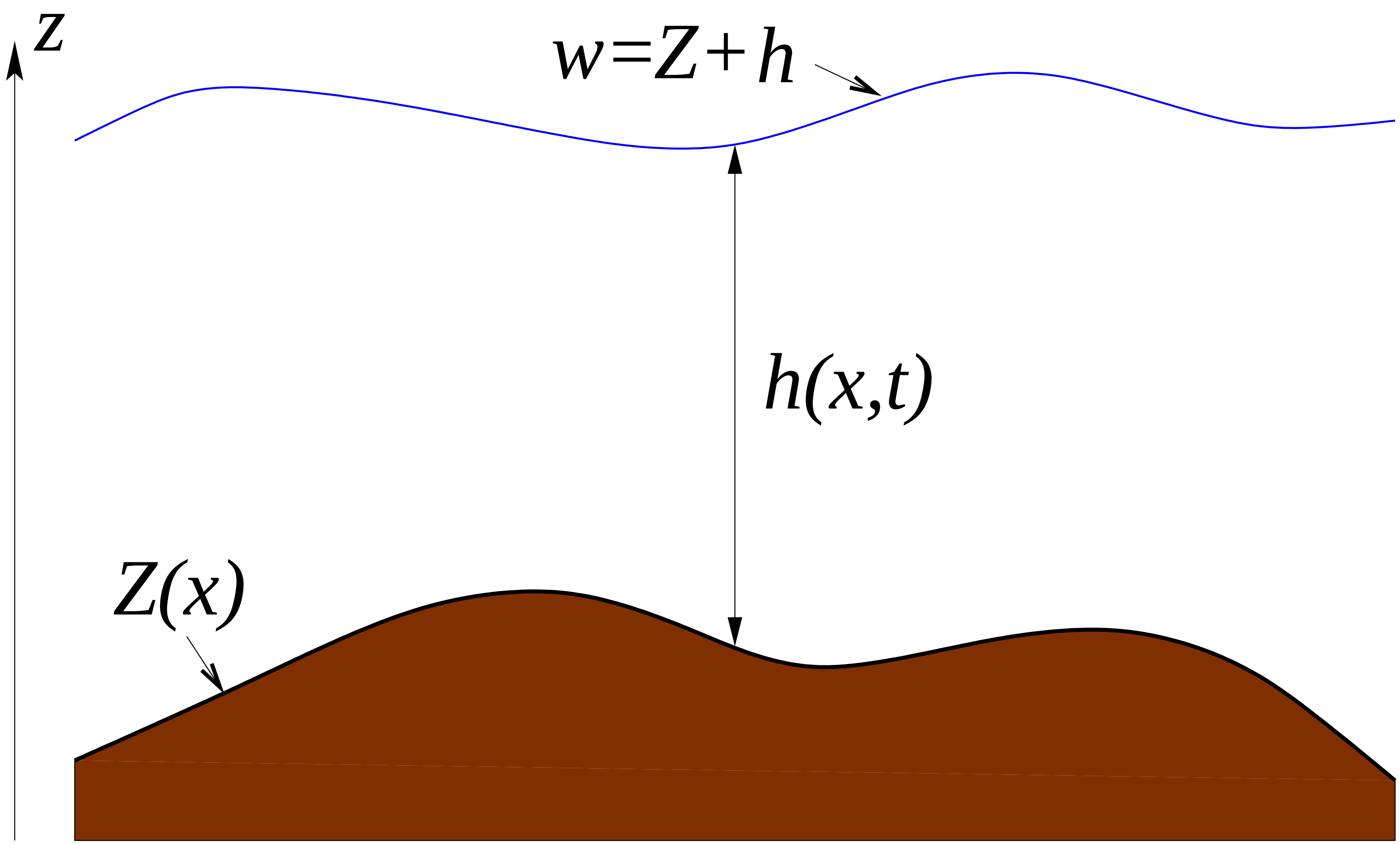
the central-upwind scheme is:

$$\frac{d}{dt}\overline{\boldsymbol{U}}_j(t) = -\frac{\boldsymbol{H}_{j+\frac{1}{2}}(t) - \boldsymbol{H}_{j-\frac{1}{2}}(t)}{\Delta x} + \boxed{\boxed{\overline{\boldsymbol{S}}_j(t)}}$$

where

$$\overline{\boldsymbol{S}}_j(t) \approx \frac{1}{\Delta x} \int\limits_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \boldsymbol{S}(x,t)\,\mathrm{d}x$$

Shallow Water Equations



1-D Saint-Venant System

$$\begin{cases} h_t + q_x = 0 \\ q_t + \left(hu^2 + \frac{g}{2}h^2\right)_x = -ghZ_x \end{cases}$$

This is a system of hyperbolic balance laws

$$\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U}, Z)_x = \boldsymbol{S}(\boldsymbol{U}, Z), \quad \boldsymbol{U} := (h, q)^\top$$

h: depth

u: velocity

$q := hu$: discharge

Z: bottom topography

g: gravitational constant

Saint-Venant System — Numerical Challenges

$$\begin{cases} h_t + q_x = 0 \\ q_t + \left(hu^2 + \frac{g}{2}h^2\right)_x = -ghZ_x \end{cases}$$

- Steady-state solutions:

$$q = \text{Const}, \quad \frac{u^2}{2} + g(h+Z) = \text{Const}$$

- "Lake at rest" steady-state solutions:

$$u = 0, \quad h + Z = \text{Const}$$

- Dry ($h = 0$) or near dry ($h \sim 0$) states

Shallow Water Equations — Naïve Source Approximation

$$\begin{cases} h_t + q_x = 0 \\ q_t + \left(hu^2 + \frac{g}{2}h^2\right)_x = -ghZ_x \end{cases}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\boldsymbol{U}}_j(t) = -\frac{\boldsymbol{H}_{j+\frac{1}{2}}(t) - \boldsymbol{H}_{j-\frac{1}{2}}(t)}{\Delta x} + \boxed{\boxed{\overline{\boldsymbol{S}}_j(t)}}, \quad \overline{\boldsymbol{U}}_j(t) := (\bar{h}_j(t), \bar{q}_j(t))^\top$$

where we use the midpoint quadrature:

$$\overline{\boldsymbol{S}}_j \approx \frac{1}{\Delta x}\int\limits_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \boldsymbol{S}(\boldsymbol{U}(x,t), Z(x))\,\mathrm{d}x \approx \boldsymbol{S}(\boldsymbol{U}_j(t), Z(x_j))$$

that is, we take

$$\overline{\boldsymbol{S}}_j = (0, -g\bar{h}_j Z_x(x_j))^T$$

Example — Small Perturbation of a Steady State

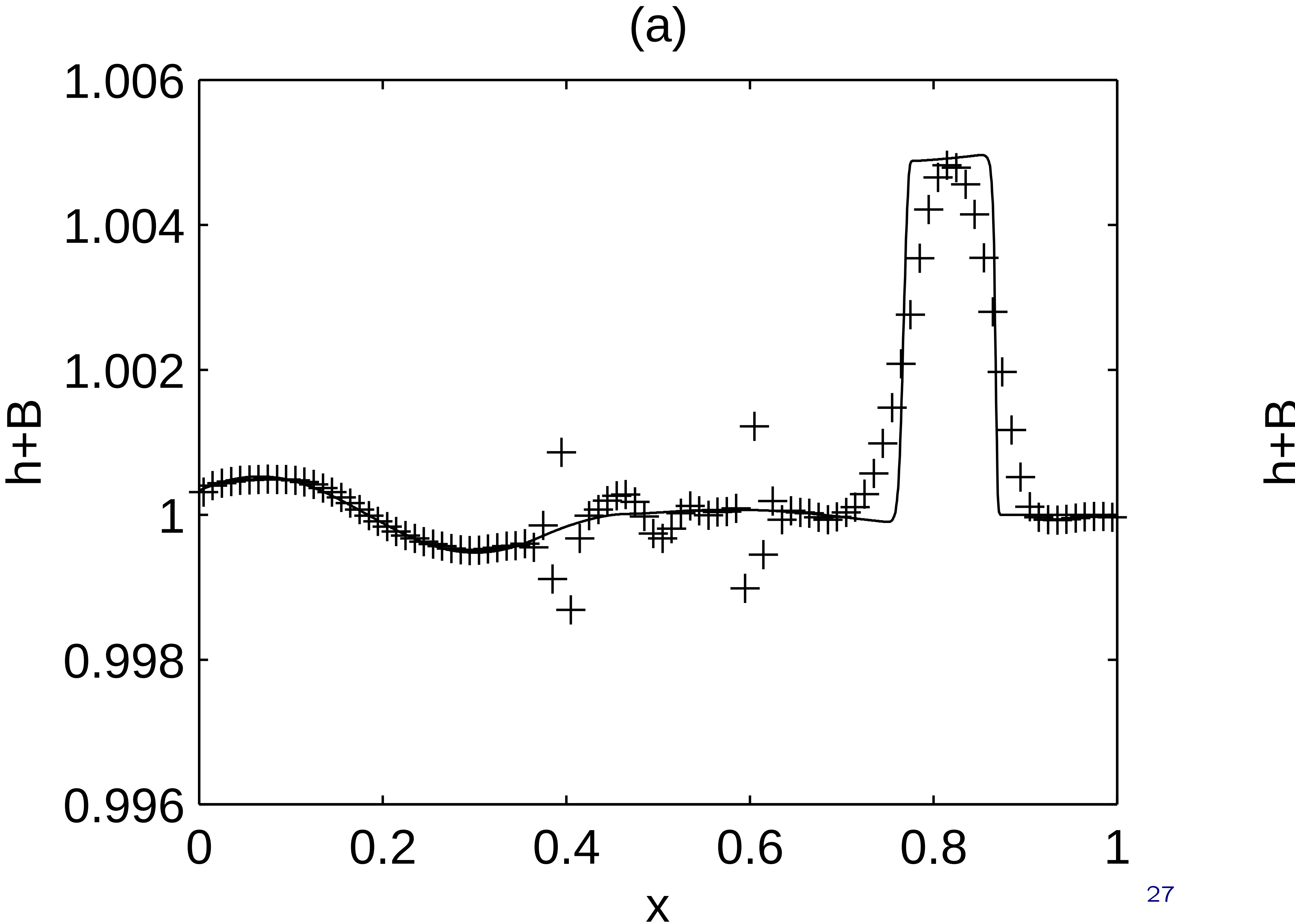
The bottom topography contains a "hump":

$$Z(x) = \begin{cases} 0.25(\cos(\pi(x-0.5)/0.1)+1), & 0.4 < x < 0.6 \\ 0, & \text{otherwise} \end{cases}$$

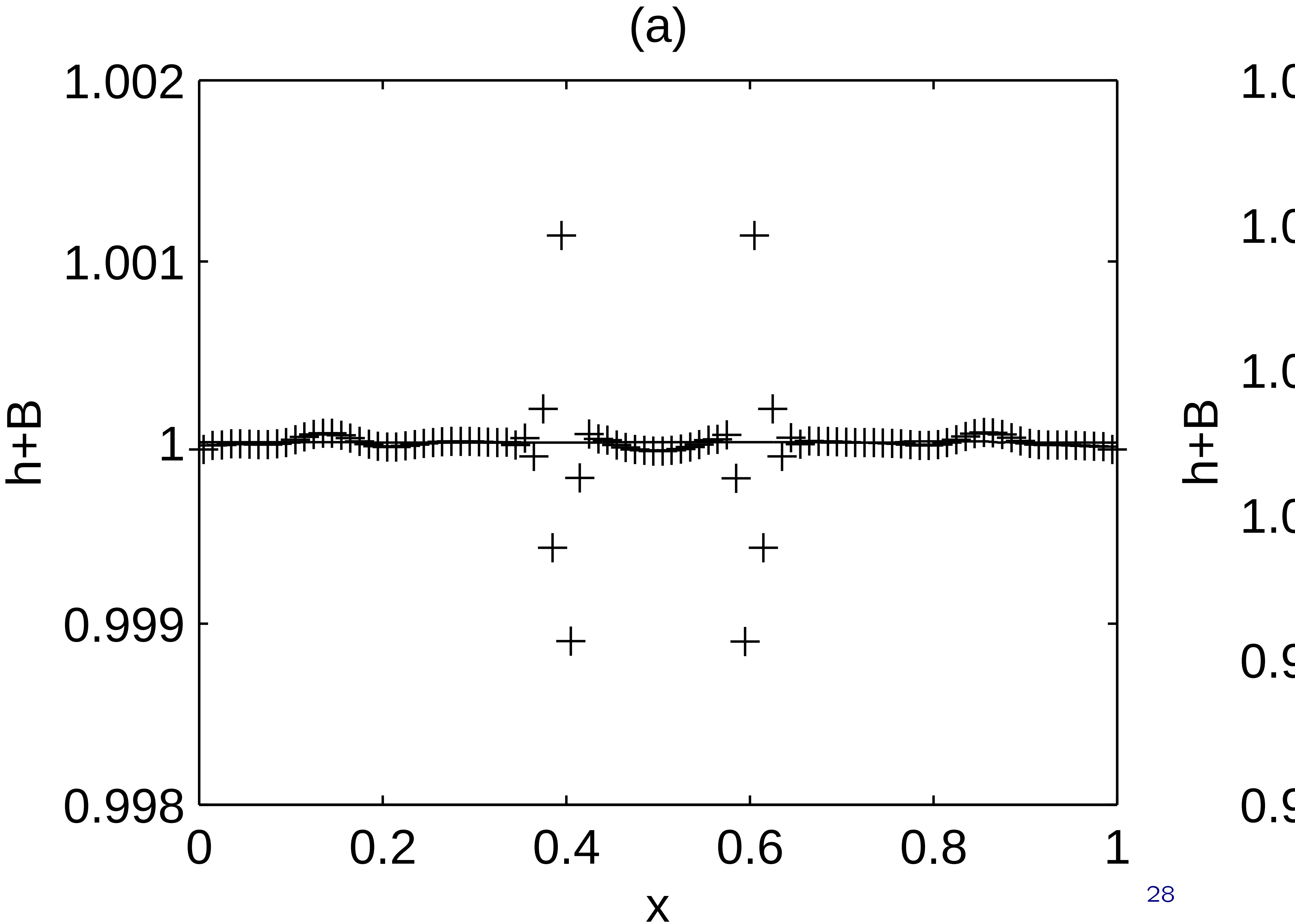
The initial data are:

$$h(x,0) + Z(x) = \begin{cases} 1+\varepsilon, & 0.1 < x < 0.2, \\ 1, & \text{otherwise}, \end{cases} \qquad u(x,0) = 0$$

$\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-5}$



(a)
1.006
1.004
1.002
1
0.998
0.996
h+B
0
0.2
0.4
0.6
0.8
1
x



(a)
1.002
1.001
1
0.999
0.998
h+B
0
0.2
0.4
0.6
0.8
1
x
h+B
1.0
1.0
1.0
1.0
0.9
0.9

Well-Balanced Central-Upwind Scheme

[Kurganov, Levy; 2002]

$w = h + Z$: water surface

$$\begin{cases} h_t + q_x = 0 \\ q_t + \left(hu^2 + \frac{g}{2}h^2\right)_x = -ghZ_x \end{cases}$$

$$\Updownarrow \qquad (h,q) \to (w,q)$$

$$\begin{cases} w_t + q_x = 0 \\ q_t + \left(\frac{q^2}{w-Z} + \frac{g}{2}(w-Z)^2\right)_x = -g(w-Z)Z_x \end{cases}$$

"Lake at rest" steady states: $\boxed{u \equiv 0,\ w \equiv \text{Const}}$

At the "lake at rest" steady state: $q = 0,\ w = \text{Const}$

$\Longrightarrow$ the flux is $\boldsymbol{F} = (q, \frac{q^2}{w-Z} + \frac{g}{2}(w-Z)^2)^\top = (0, \frac{g}{2}(w-Z)^2)^\top$

$\Longrightarrow$ the second component of the numerical flux is

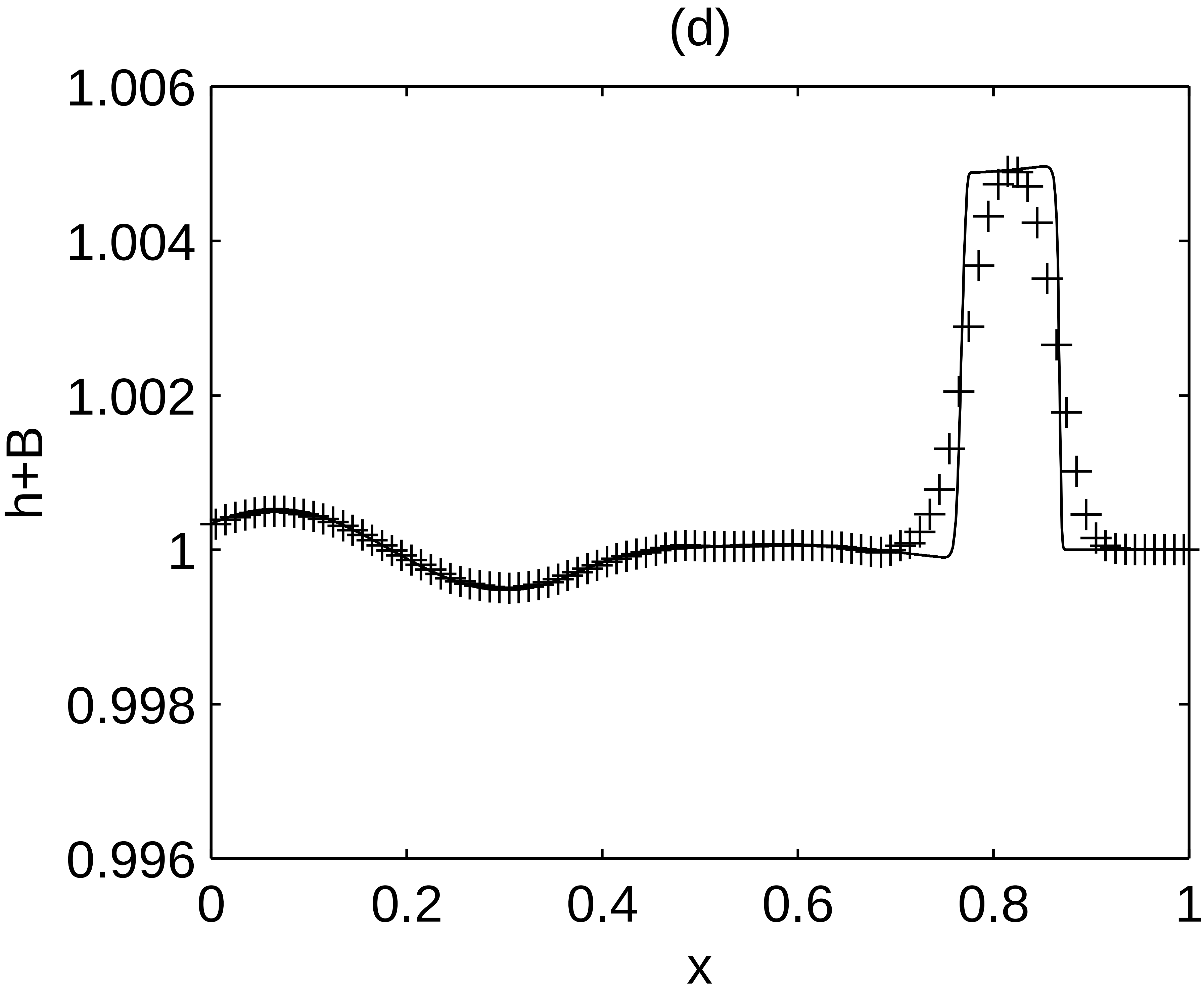
$$\boldsymbol{H}^{(2)}_{j+\frac{1}{2}} = \frac{g}{2}\left(w - Z(x_{j+\frac{1}{2}})\right)^2, \quad \boldsymbol{H}^{(2)}_{j-\frac{1}{2}} = \frac{g}{2}\left(w - Z(x_{j-\frac{1}{2}})\right)^2$$

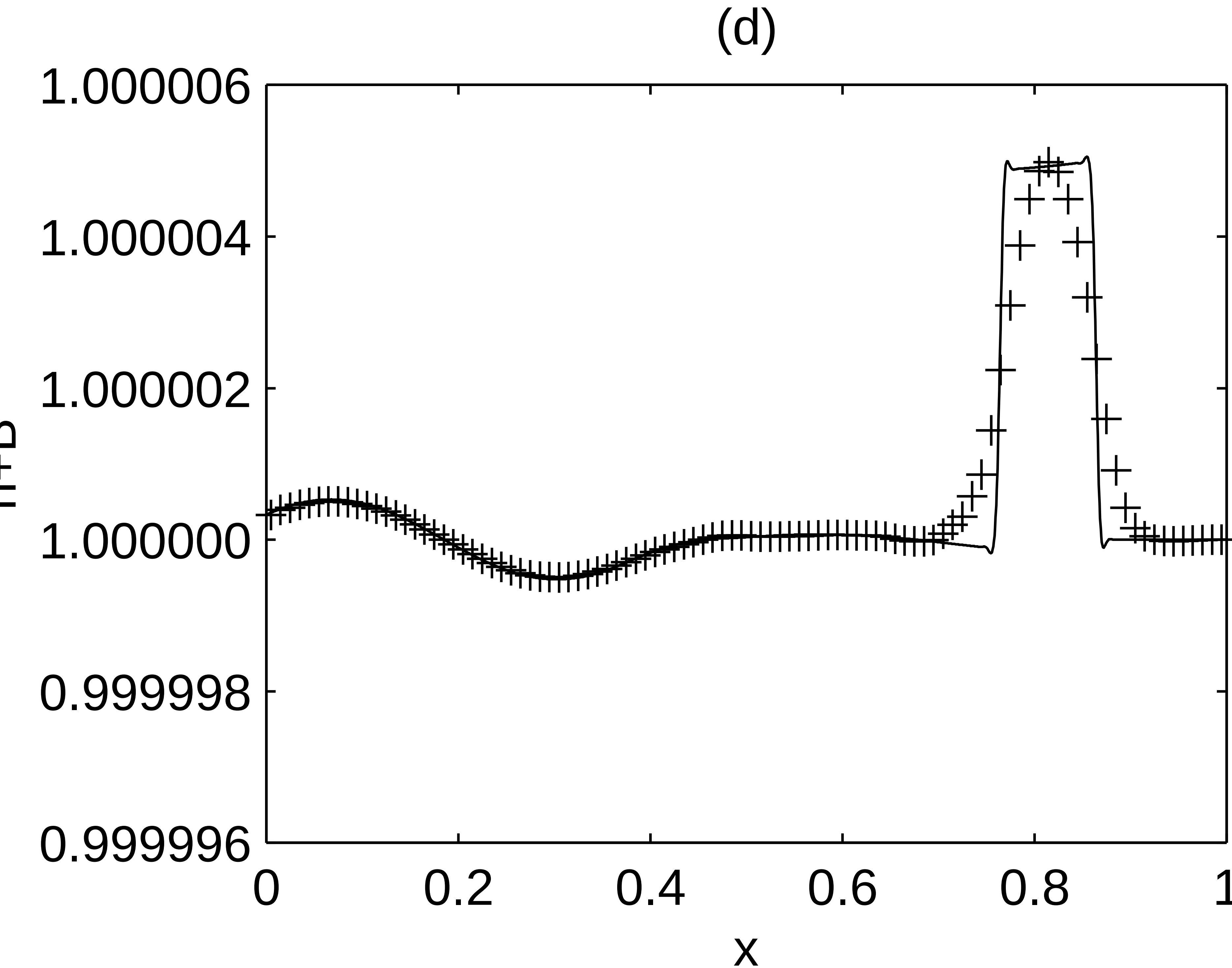
$$\Longrightarrow \frac{d}{dt}\bar{q}_j(t) = -\frac{\boldsymbol{H}^{(2)}_{j+\frac{1}{2}}(t) - \boldsymbol{H}^{(2)}_{j-\frac{1}{2}}(t)}{\Delta x} + \overline{\boldsymbol{S}}^{(2)}_j(t)$$

$$= g \cdot \frac{Z(x_{j+\frac{1}{2}}) - Z(x_{j-\frac{1}{2}})}{\Delta x} \cdot \frac{(w - Z(x_{j+\frac{1}{2}})) + (w - Z(x_{j-\frac{1}{2}}))}{2} + \overline{\boldsymbol{S}}^{(2)}_j(t)$$

$\Longrightarrow$ The well-balanced quadrature is

$$\overline{\boldsymbol{S}}^{(2)}_j(t) = -g \cdot \frac{Z(x_{j+\frac{1}{2}}) - Z(x_{j-\frac{1}{2}})}{\Delta x} \cdot \left(\overline{w}_j - \frac{Z(x_{j+\frac{1}{2}}) + Z(x_{j-\frac{1}{2}})}{2}\right)$$

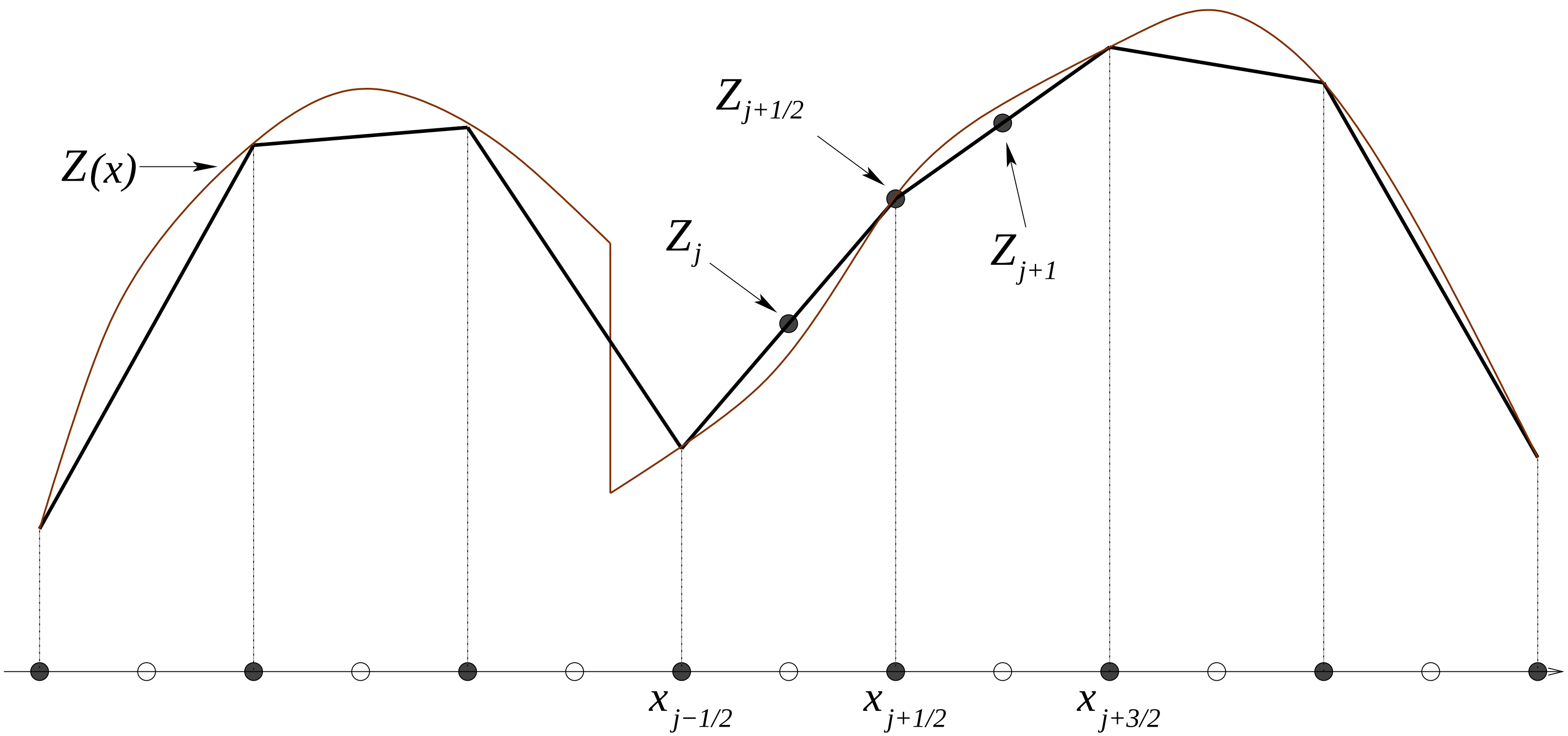

(d)
1.006
1.004
1.002
1
0.998
0.996
h+B
0
0.2
0.4
0.6
0.8
1
x


(d)
1.000006
1.000004
1.000002
1.000000
0.999998
0.999996
0
0.2
0.4
0.6
0.8
1
x

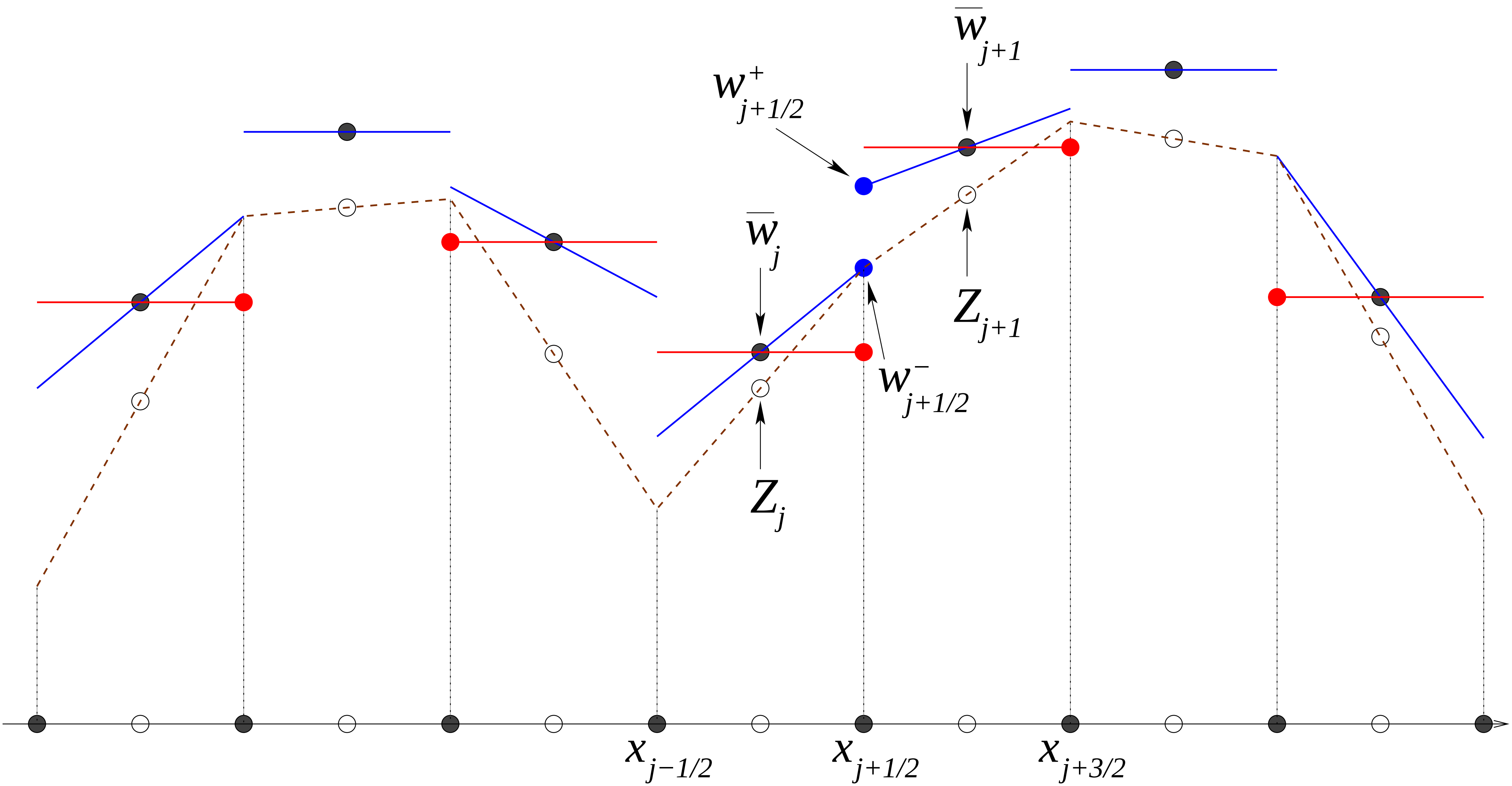
Well-Balanced Positivity Preserving Central-Upwind Scheme

[Kurganov, Petrova; 2007]

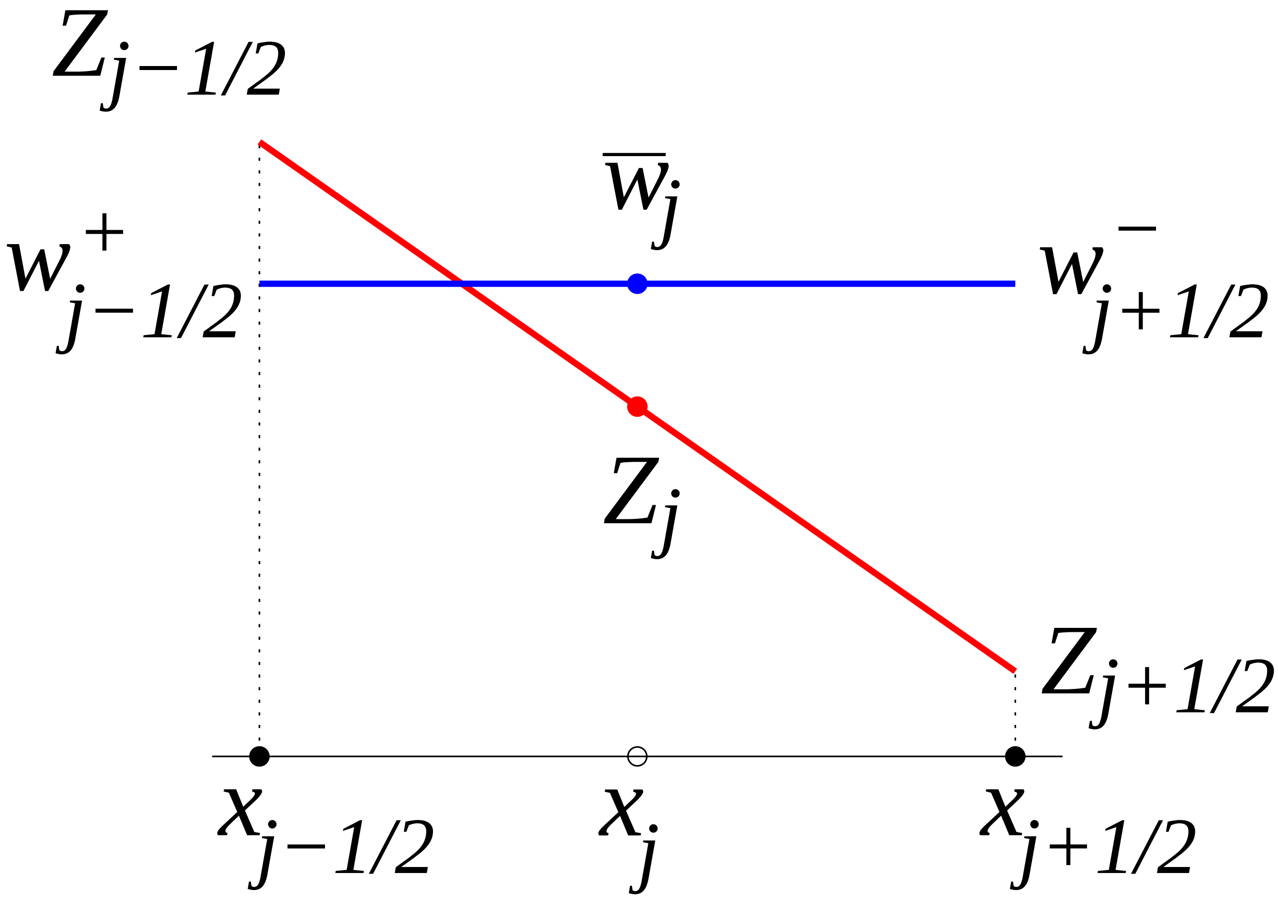
Step 1: Piecewise linear reconstruction of the bottom

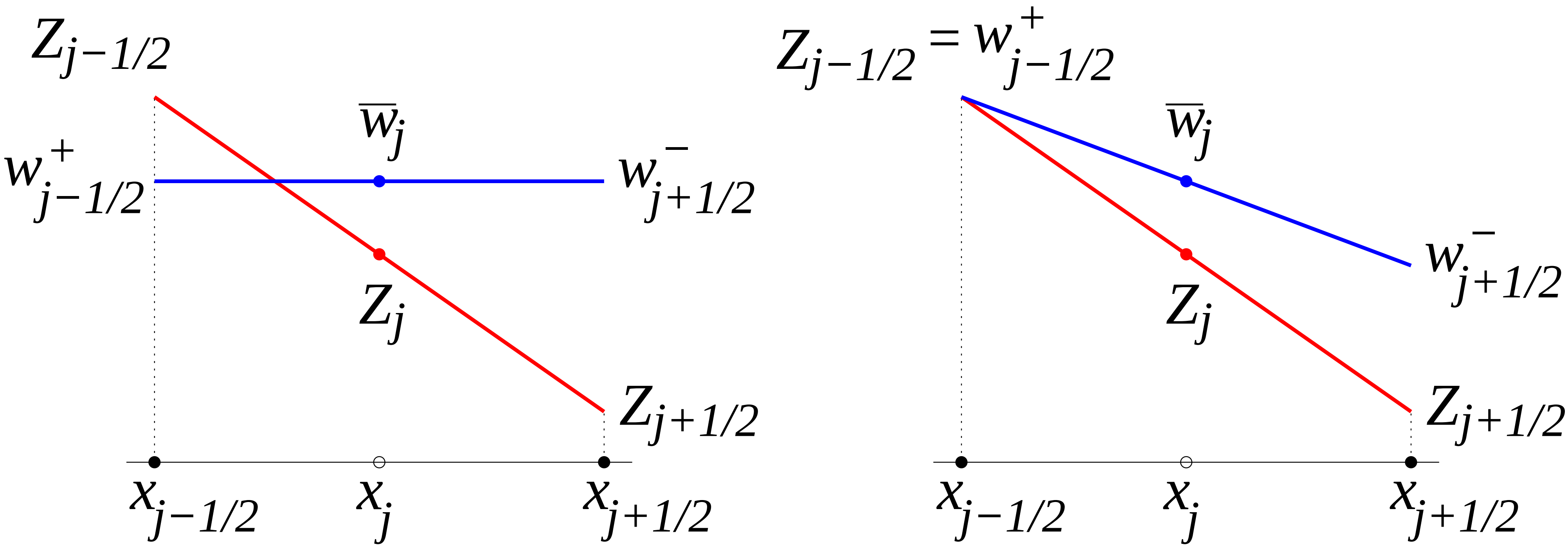


Step 2: Positivity preserving reconstruction of w



$$h^{\pm}_{j+\frac{1}{2}} = \boxed{w^{\pm}_{j+\frac{1}{2}} - Z_{j+\frac{1}{2}}}$$





$Z_{j-1/2}$
$\overline{w}_j$
$w^+_{j-1/2}$
$w^-_{j+1/2}$
Z_j
$Z_{j+1/2}$
$x_{j-1/2}$
x_j
$x_{j+1/2}$
$Z_{j-1/2} = w^+_{j-1/2}$
$\overline{w}_j$
$w^-_{j+1/2}$
Z_j
$Z_{j+1/2}$
$x_{j-1/2}$
x_j
$x_{j+1/2}$

Step 3: Desingularization $\left(u \neq \dfrac{q}{h}\right.$ for small $\left.h\right)$

– Simplest:

$$u = \begin{cases} \dfrac{q}{h}, & \text{if } h \geq \varepsilon \\ 0, & \text{if } h < \varepsilon \end{cases}$$

– More sophisticated (smoother transition for small h):

$$u = \frac{2\,h\,q}{h^2 + \max(h^2, \varepsilon)} \quad \text{or} \quad u = \frac{\sqrt{2}\,h\,q}{\sqrt{h^4 + \max(h^4, \varepsilon)}}$$

Remark: For consistency, one has to recompute the discharge:

$$q = h \cdot u$$

Positivity Preserving Property

If an SSP ODE solver is used, then

$$\bar{h}_j^{n+1} = \alpha^-_{j-\frac{1}{2}} h^-_{j-\frac{1}{2}} + \alpha^+_{j-\frac{1}{2}} h^+_{j-\frac{1}{2}} + \alpha^-_{j+\frac{1}{2}} h^-_{j+\frac{1}{2}} + \alpha^+_{j+\frac{1}{2}} h^+_{j+\frac{1}{2}}$$

where the coefficients $\alpha^{\pm}_{j\pm\frac{1}{2}} > 0$ provided an appropriate CFL condition is satisfied:

- 1-D: CFL number is 1/2
- 2-D Cartesian mesh: CFL number is 1/4
- 2-D triangular mesh: CFL number is 1/3

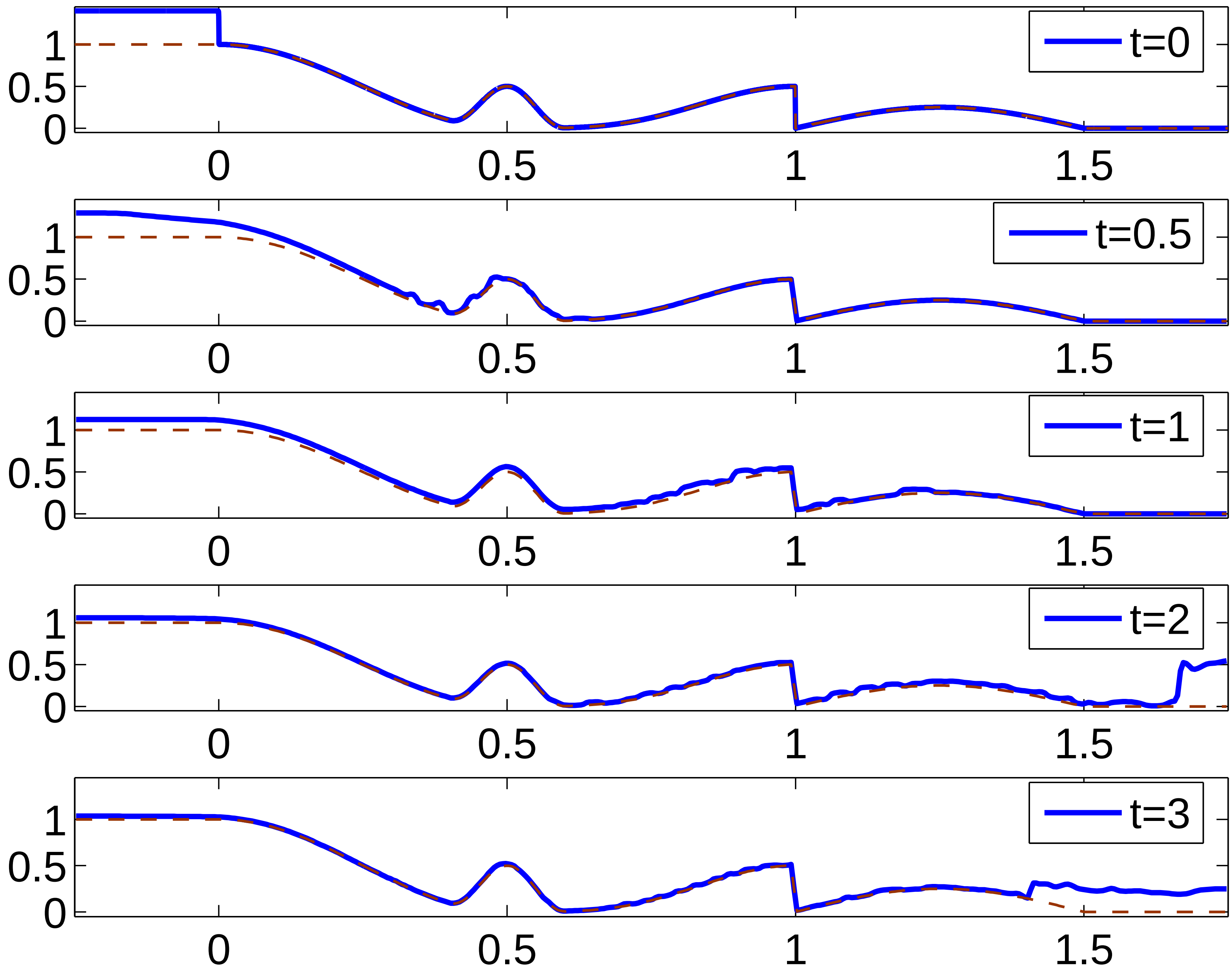
Remark: For high-order SSP methods, adaptive timestep control has to be implemented.

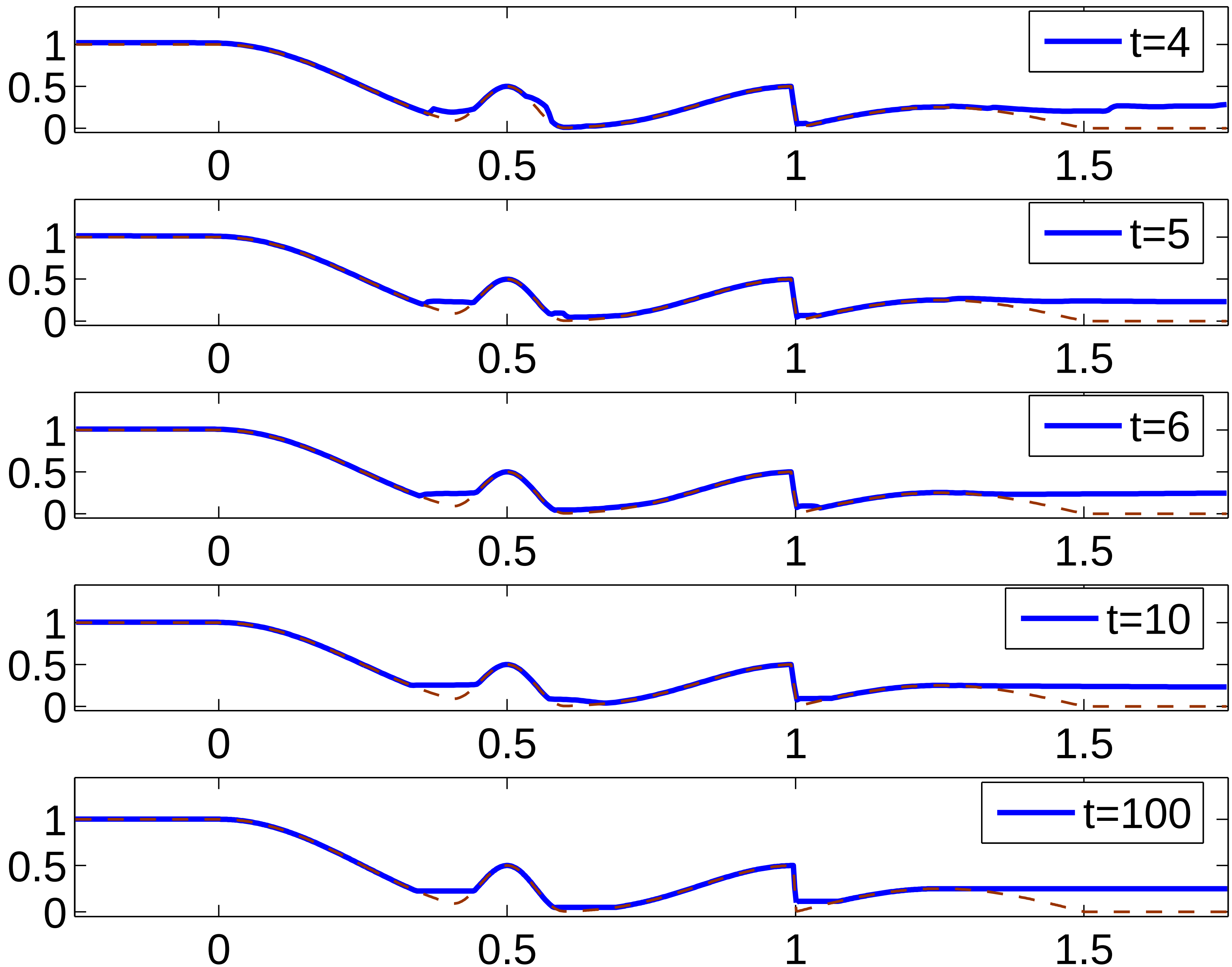
Example — ShW with Friction and Discontinuous Bottom

$$\begin{cases} h_t + q_x = 0 \\ q_t + \left(hu^2 + \frac{g}{2}h^2\right)_x = -ghZ_x - \boxed{\boxed{\kappa(h)u}}, \end{cases} \qquad \kappa(h) = \frac{0.001}{1+10h}$$

$$Z(x) = \begin{cases} 1, & x < 0 \\ \cos^2(\pi x), & 0 \le x \le 0.4 \\ \cos^2(\pi x) + 0.25(\cos(10\pi(x-0.5)) + 1), & 0.4 \le x \le 0.5 \\ 0.5\cos^4(\pi x) + 0.25(\cos(10\pi(x-0.5)) + 1), & 0.5 \le x \le 0.6 \\ 0.5\cos^4(\pi x), & 0.5 \le x < 1 \\ 0.25\sin(2\pi(x-1)), & 1 < x \le 1.5 \\ 0, & x > 1.5. \end{cases}$$

$$(w(x,0), u(x,0)) = \begin{cases} (1.4, 0), & x < 0 \\ (Z(x), 0), & x > 0 \end{cases} \qquad \text{(Dam break)}$$


t=0
t=0.5
t=1
t=2
t=3
1
0.5
0
0
0.5
1
1.5


t=4
t=5
t=6
t=10
t=100
0
0.5
1
1.5
0
0.5
1

Central-Upwind Schemes for the 2-D Saint-Venant System

Cartesian Grid: [Kurganov, Levy, 2002], [Kurganov, Petrova; 2007]

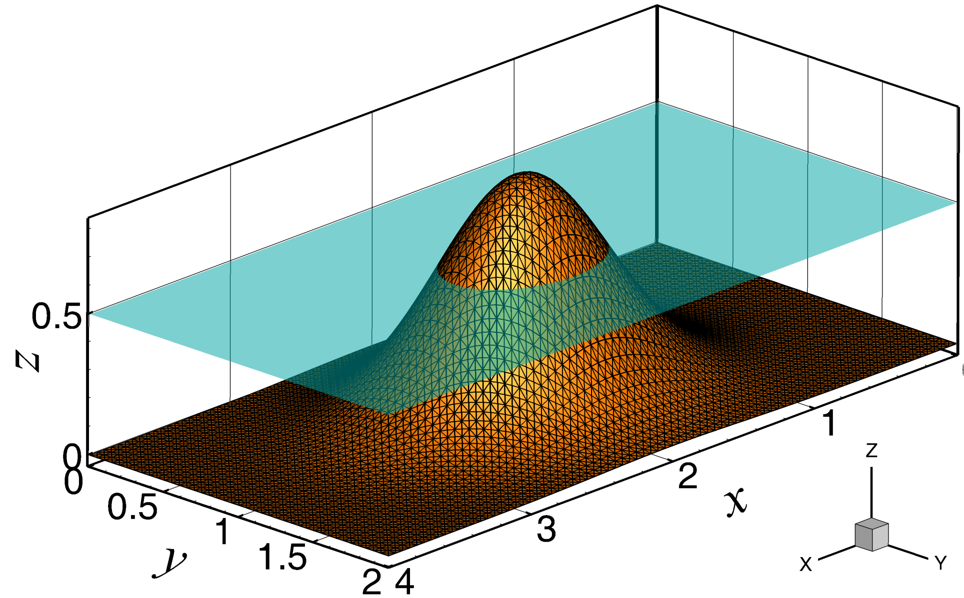
Triangular Grid: [Bryson, Epshteyn, Kurganov, Petrova; 2011],

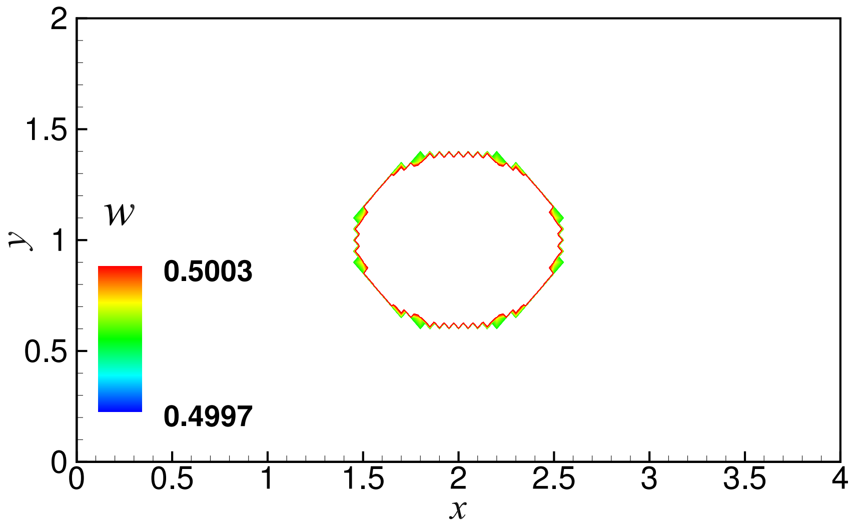
[Liu, Albright, Epshteyn, Kurganov; 2018]

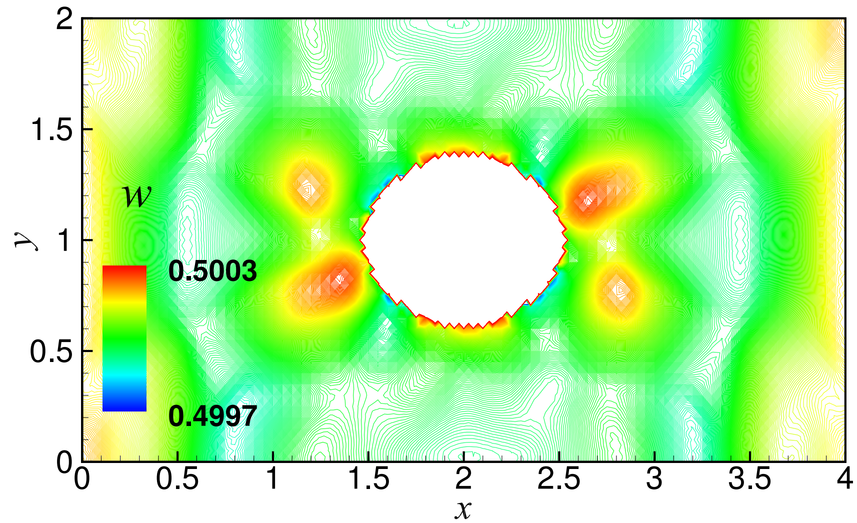
Unstructured Quadrilateral Mesh:
[Shirkhani, Mohammadian, Seidou, Kurganov; 2016]

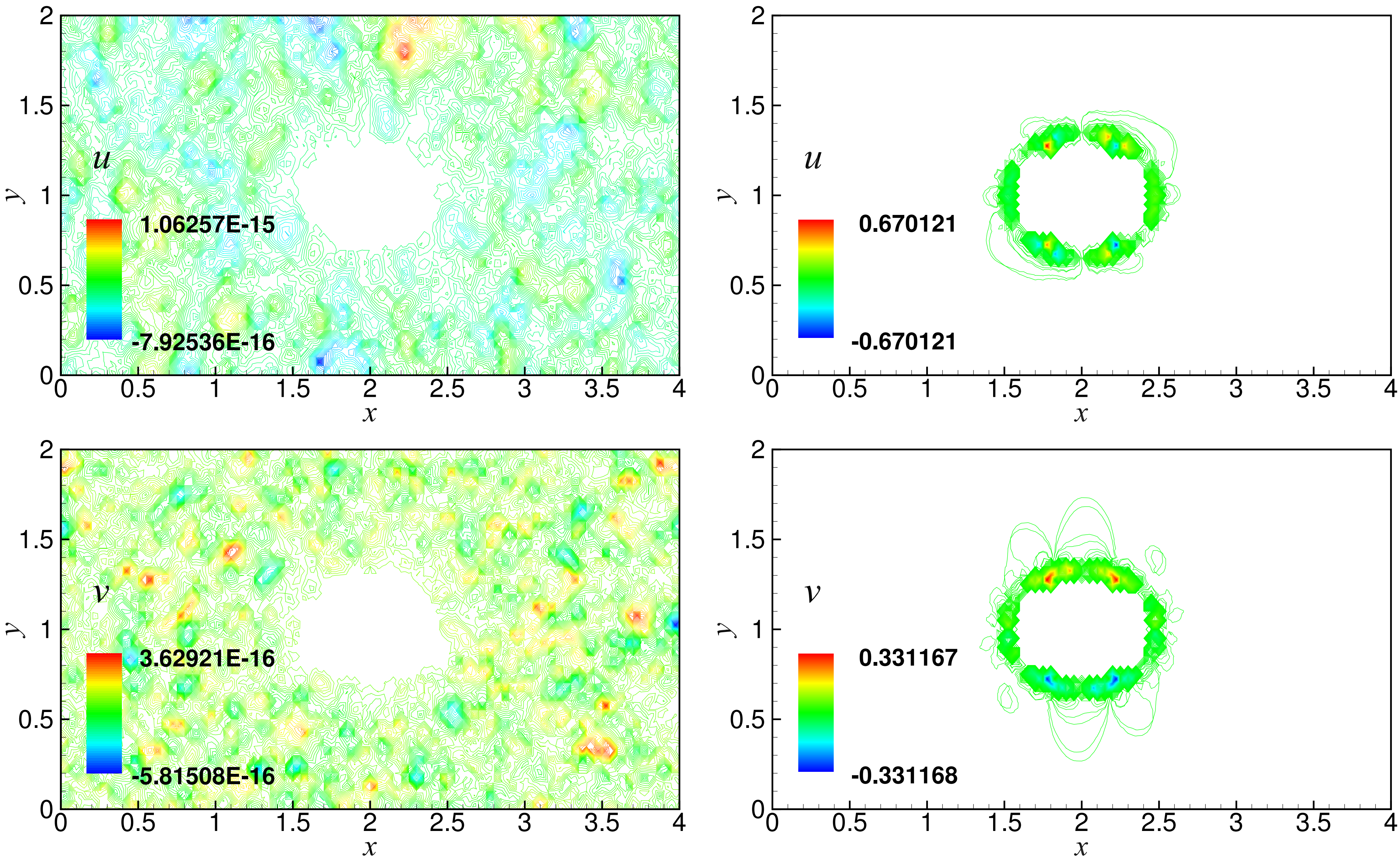
Polygonal Cell-Vertex Mesh:
[Beljadid, Mohammadian, Kurganov; 2016]

Example — "Lake at Rest" Steady State in the Domain with Wet/Dry Interfaces



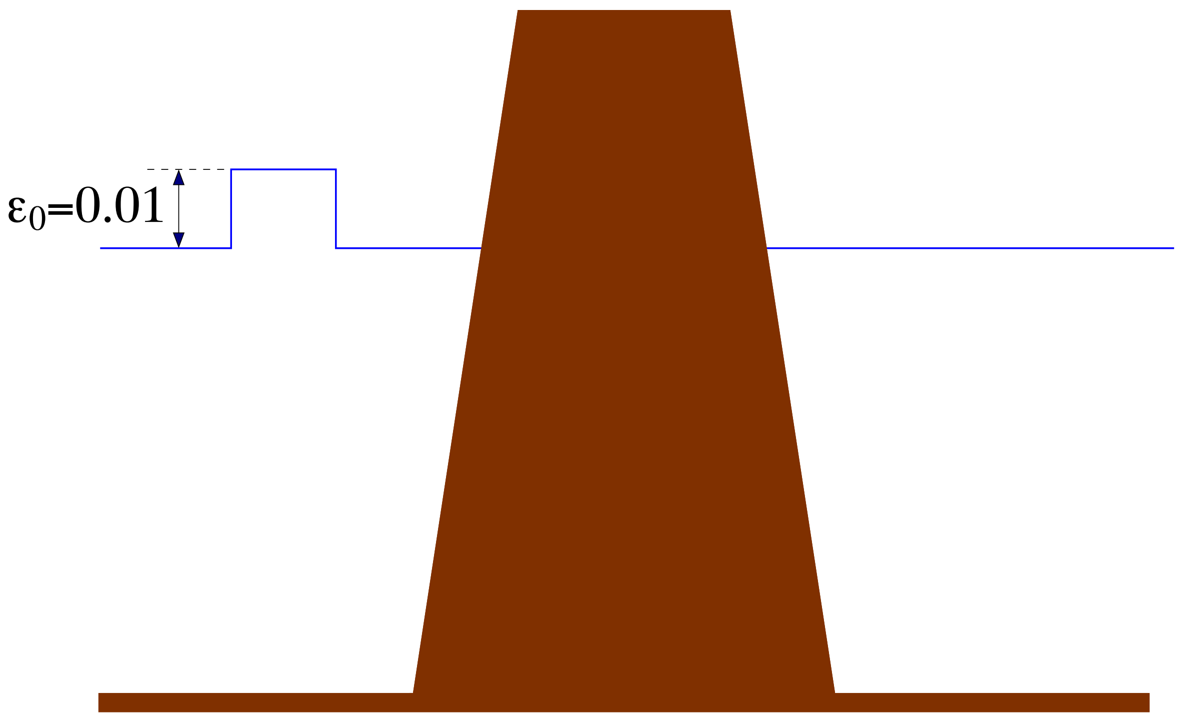




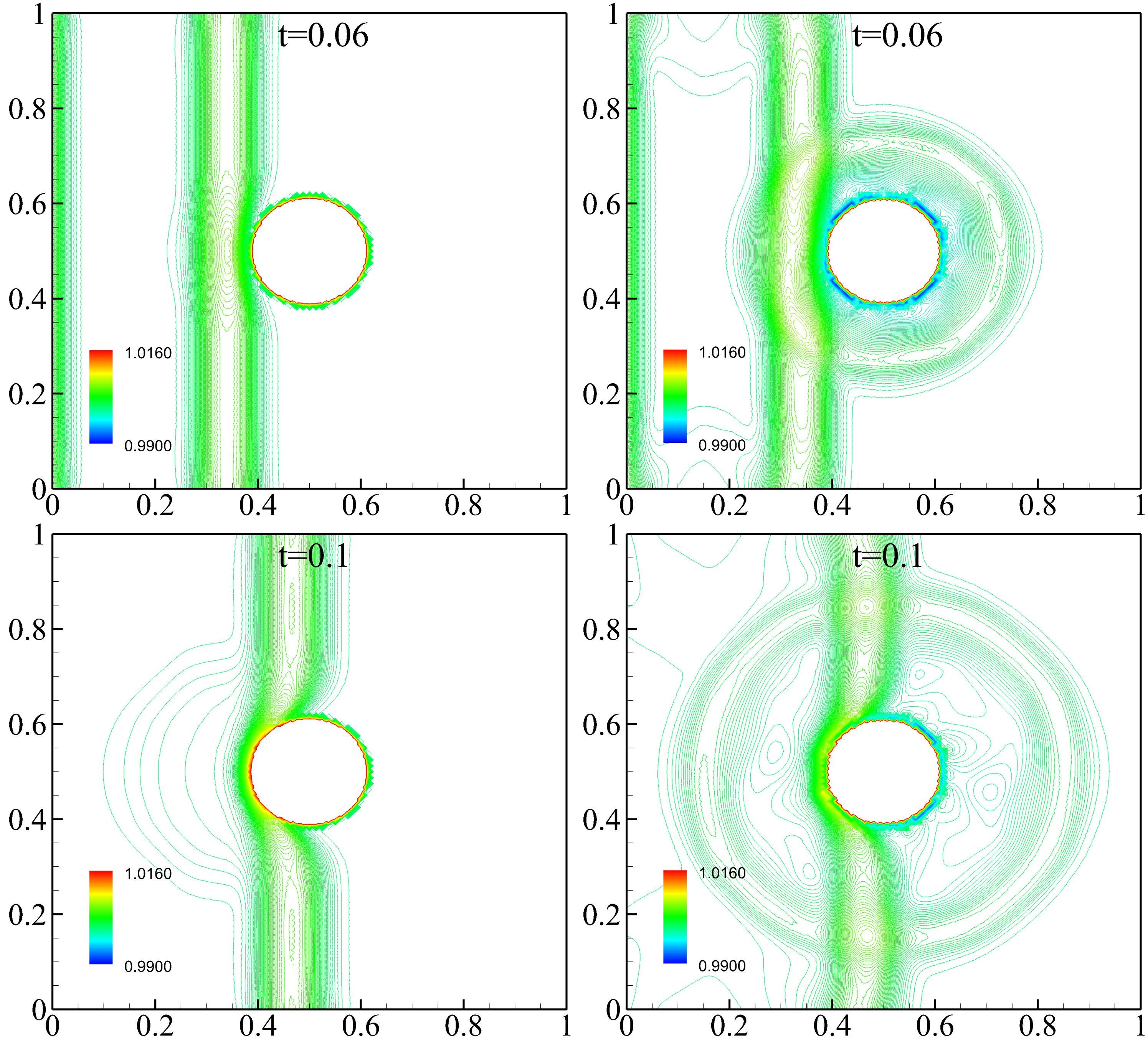


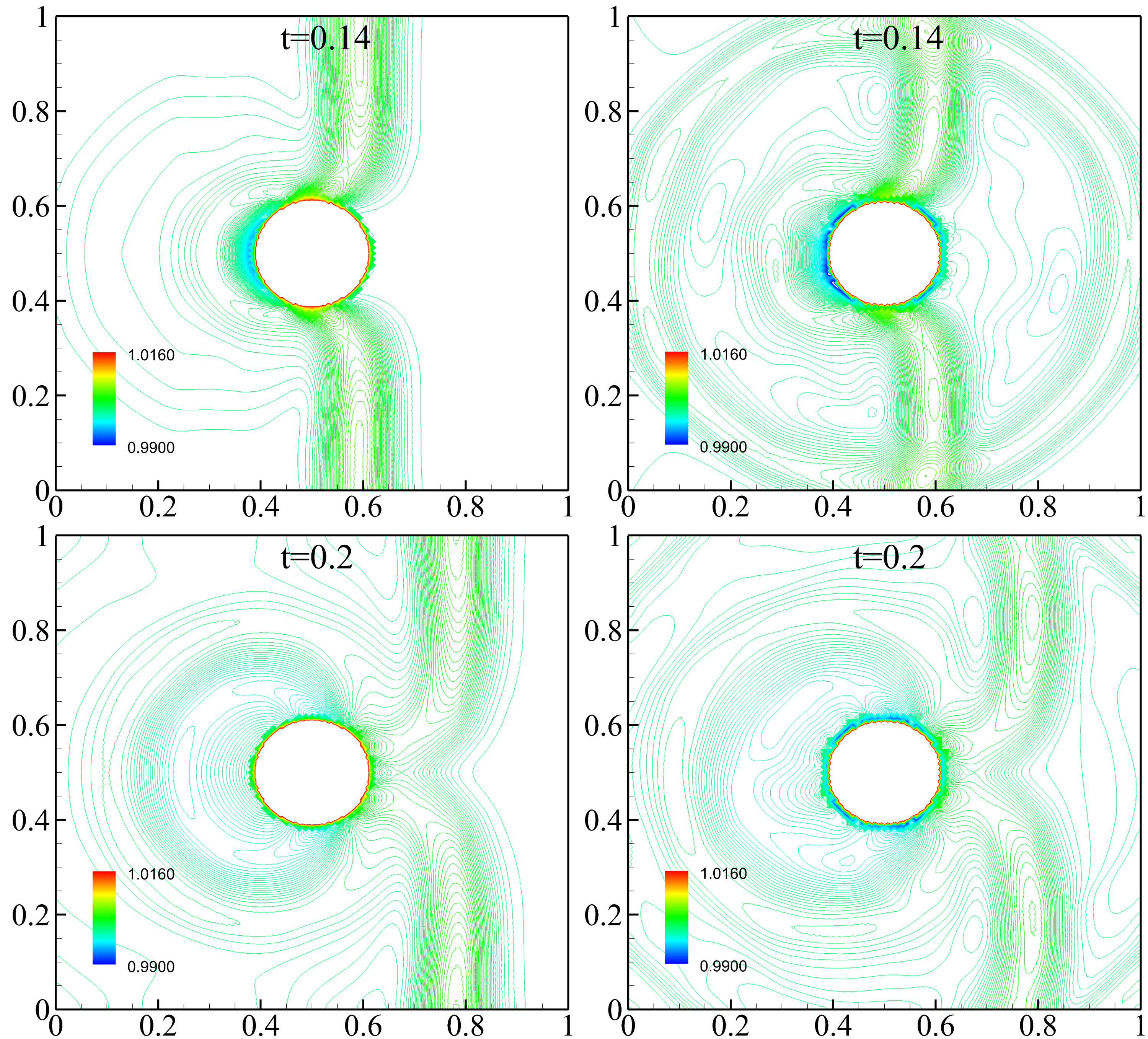
u
1.06257E-15
-7.92536E-16
u
0.670121
-0.670121
v
3.62921E-16
-5.81508E-16
v
0.331167
-0.331168
x
y

Example — Small Perturbation of "Lake at Rest" Steady State



1-D slice of the bottom topography and water surface (not to scale)


t=0.06
t=0.06
t=0.1
t=0.1
1.0160
0.9900
0
0.2
0.4
0.6
0.8
1



t=0.14
t=0.14
t=0.2
t=0.2
1.0160
0.9900
0
0.2
0.4
0.6
0.8
1

Path-Conservative Central-Upwind Schemes for Nonconservative Hyperbolic Systems

[Castro Díaz, Morales de Luna, Kurganov; preprint]

Reformulated Central-Upwind Scheme

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\boldsymbol{U}}_j(t) = -\frac{\boldsymbol{H}_{j+\frac{1}{2}}(t) - \boldsymbol{H}_{j-\frac{1}{2}}(t)}{\Delta x}$$

$$\boldsymbol{H}_{j+\frac{1}{2}} = \frac{a^+_{j+\frac{1}{2}}\boldsymbol{F}(\boldsymbol{U}^-_{j+\frac{1}{2}}) - a^-_{j+\frac{1}{2}}\boldsymbol{F}(\boldsymbol{U}^+_{j+\frac{1}{2}})}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} + \frac{a^+_{j+\frac{1}{2}}a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}\left(\boldsymbol{U}^+_{j+\frac{1}{2}} - \boldsymbol{U}^-_{j+\frac{1}{2}}\right)$$

Define the following two coefficients:

$$\alpha_0^{j+\frac{1}{2}} := \frac{-2\,a^+_{j+\frac{1}{2}}a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}, \qquad \alpha_1^{j+\frac{1}{2}} := \frac{a^+_{j+\frac{1}{2}} + a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}$$

Then the central-upwind flux can be rewritten as

$$\boldsymbol{H}_{j+\frac{1}{2}} = \frac{1-\alpha_1^{j+\frac{1}{2}}}{2}\boldsymbol{F}(\boldsymbol{U}^+_{j+\frac{1}{2}}) + \frac{1+\alpha_1^{j+\frac{1}{2}}}{2}\boldsymbol{F}(\boldsymbol{U}^-_{j+\frac{1}{2}}) - \frac{\alpha_0^{j+\frac{1}{2}}}{2}\Big(\boldsymbol{U}^+_{j+\frac{1}{2}} - \boldsymbol{U}^-_{j+\frac{1}{2}}\Big)$$

We then compute the differences between the numerical flux and the physical fluxes at both sides of the cell interface:

$$\boldsymbol{D}^-_{j+\frac{1}{2}} := \boldsymbol{H}_{j+\frac{1}{2}} - \boldsymbol{F}(\boldsymbol{U}^-_{j+\frac{1}{2}}) = \frac{1}{2}\Big[\big(1-\alpha_1^{j+\frac{1}{2}}\big)\Big(\boldsymbol{F}(\boldsymbol{U}^+_{j+\frac{1}{2}}) - \boldsymbol{F}(\boldsymbol{U}^-_{j+\frac{1}{2}})\Big) - \alpha_0^{j+\frac{1}{2}}\Big(\boldsymbol{U}^+_{j+\frac{1}{2}} - \boldsymbol{U}^-_{j+\frac{1}{2}}\Big)\Big]$$

$$\boldsymbol{D}^+_{j+\frac{1}{2}} := \boldsymbol{F}(\boldsymbol{U}^+_{j+\frac{1}{2}}) - \boldsymbol{H}_{j+\frac{1}{2}} = \frac{1}{2}\Big[\big(1+\alpha_1^{j+\frac{1}{2}}\big)\Big(\boldsymbol{F}(\boldsymbol{U}^+_{j+\frac{1}{2}}) - \boldsymbol{F}(\boldsymbol{U}^-_{j+\frac{1}{2}})\Big) + \alpha_0^{j+\frac{1}{2}}\Big(\boldsymbol{U}^+_{j+\frac{1}{2}} - \boldsymbol{U}^-_{j+\frac{1}{2}}\Big)\Big]$$

and rewrite the semi-discrete central-upwind scheme as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\overline{\boldsymbol{U}}_j &= -\frac{1}{\Delta x}\Big(\boldsymbol{H}_{j+\frac{1}{2}} - \boldsymbol{H}_{j-\frac{1}{2}}\Big) \\
&= -\frac{1}{\Delta x}\Big(\boldsymbol{H}_{j+\frac{1}{2}} - \boldsymbol{F}(\boldsymbol{U}^-_{j+\frac{1}{2}}) + \boldsymbol{F}(\boldsymbol{U}^+_{j-\frac{1}{2}}) - \boldsymbol{H}_{j-\frac{1}{2}} + \boldsymbol{F}(\boldsymbol{U}^-_{j+\frac{1}{2}}) - \boldsymbol{F}(\boldsymbol{U}^+_{j-\frac{1}{2}})\Big) \\
&= -\frac{1}{\Delta x}\Big(\boldsymbol{D}^+_{j-\frac{1}{2}} + \boldsymbol{D}^-_{j+\frac{1}{2}} + \boldsymbol{F}(\boldsymbol{U}^-_{j+\frac{1}{2}}) - \boldsymbol{F}(\boldsymbol{U}^+_{j-\frac{1}{2}})\Big) \\
&= -\frac{1}{\Delta x}\Big(\boldsymbol{D}^+_{j-\frac{1}{2}} + \boldsymbol{D}^-_{j+\frac{1}{2}} + \int\limits_{C_j} A(\boldsymbol{P}_j(x))\,\frac{\mathrm{d}\boldsymbol{P}_j}{\mathrm{d}x}\,\mathrm{d}x\Big)
\end{aligned}$$

Finally, we consider a sufficiently smooth path

$$\boldsymbol{\Psi}_{j+\frac{1}{2}}(s) := \boldsymbol{\Psi}(s; \boldsymbol{U}^-_{j+\frac{1}{2}}, \boldsymbol{U}^+_{j+\frac{1}{2}}), \quad s \in [0,1]$$

connecting the states $\boldsymbol{U}^-_{j+\frac{1}{2}}$ and $\boldsymbol{U}^+_{j+\frac{1}{2}}$:

$$\boldsymbol{\Psi}(0; \boldsymbol{U}^-_{j+\frac{1}{2}}, \boldsymbol{U}^+_{j+\frac{1}{2}}) = \boldsymbol{U}^-_{j+\frac{1}{2}}, \quad \boldsymbol{\Psi}(1; \boldsymbol{U}^-_{j+\frac{1}{2}}, \boldsymbol{U}^+_{j+\frac{1}{2}}) = \boldsymbol{U}^+_{j+\frac{1}{2}}$$

and then

$$\boldsymbol{D}^-_{j+\frac{1}{2}} := \boldsymbol{H}_{j+\frac{1}{2}} - \boldsymbol{F}(\boldsymbol{U}^-_{j+\frac{1}{2}}) = \frac{1}{2}\Big[\big(1 - \alpha_1^{j+\frac{1}{2}}\big)\Big(\boldsymbol{F}(\boldsymbol{U}^+_{j+\frac{1}{2}}) - \boldsymbol{F}(\boldsymbol{U}^-_{j+\frac{1}{2}})\Big) - \alpha_0^{j+\frac{1}{2}}\Big(\boldsymbol{U}^+_{j+\frac{1}{2}} - \boldsymbol{U}^-_{j+\frac{1}{2}}\Big)\Big]$$

$$\boldsymbol{D}^+_{j+\frac{1}{2}} := \boldsymbol{F}(\boldsymbol{U}^+_{j+\frac{1}{2}}) - \boldsymbol{H}_{j+\frac{1}{2}} = \frac{1}{2}\Big[\big(1 + \alpha_1^{j+\frac{1}{2}}\big)\Big(\boldsymbol{F}(\boldsymbol{U}^+_{j+\frac{1}{2}}) - \boldsymbol{F}(\boldsymbol{U}^-_{j+\frac{1}{2}})\Big) + \alpha_0^{j+\frac{1}{2}}\Big(\boldsymbol{U}^+_{j+\frac{1}{2}} - \boldsymbol{U}^-_{j+\frac{1}{2}}\Big)\Big]$$

can be written as

$$\boldsymbol{D}^\pm_{j+\frac{1}{2}} = \frac{1 \pm \alpha_1^{j+\frac{1}{2}}}{2} \int_0^1 A(\boldsymbol{\Psi}_{j+\frac{1}{2}}(s)) \frac{\mathrm{d}\boldsymbol{\Psi}_{j+\frac{1}{2}}}{\mathrm{d}s}\,\mathrm{d}s \pm \frac{\alpha_0^{j+\frac{1}{2}}}{2}\Big(\boldsymbol{U}^+_{j+\frac{1}{2}} - \boldsymbol{U}^-_{j+\frac{1}{2}}\Big)$$

Reformulated Central-Upwind Scheme (summary)

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\boldsymbol{U}}_j(t) = -\frac{1}{\Delta x}\Big(\boldsymbol{D}^+_{j-\frac{1}{2}} + \boldsymbol{D}^-_{j+\frac{1}{2}} + \int\limits_{C_j} A(\boldsymbol{P}_j(x))\,\frac{\mathrm{d}\boldsymbol{P}_j}{\mathrm{d}x}\,\mathrm{d}x\Big)$$

$$\boldsymbol{D}^\pm_{j+\frac{1}{2}} = \frac{1 \pm \alpha_1^{j+\frac{1}{2}}}{2}\int\limits_0^1 A(\boldsymbol{\Psi}_{j+\frac{1}{2}}(s))\,\frac{\mathrm{d}\boldsymbol{\Psi}_{j+\frac{1}{2}}}{\mathrm{d}s}\,\mathrm{d}s \pm \frac{\alpha_0^{j+\frac{1}{2}}}{2}\Big(\boldsymbol{U}^+_{j+\frac{1}{2}} - \boldsymbol{U}^-_{j+\frac{1}{2}}\Big)$$

$$\alpha_0^{j+\frac{1}{2}} := \frac{-2\,a^+_{j+\frac{1}{2}}a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}, \qquad \alpha_1^{j+\frac{1}{2}} := \frac{a^+_{j+\frac{1}{2}} + a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}$$

Nonconservative Hyperbolic Systems

$$\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U})_x = B(\boldsymbol{U})\boldsymbol{U}_x$$

Quasilinear form:

$$\boldsymbol{U}_t + \mathcal{A}(\boldsymbol{U})\boldsymbol{U}_x = \boldsymbol{0}, \quad \mathcal{A}(\boldsymbol{U}) := \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}(\boldsymbol{U}) - B(\boldsymbol{U})$$

The reformulated semi-discrete central-upwind scheme can be directly generalized to this quasilinear system replacing A with $\mathcal{A}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\boldsymbol{U}}_j = -\frac{1}{\Delta x}\Big(\boldsymbol{D}^+_{j-\frac{1}{2}} + \boldsymbol{D}^-_{j+\frac{1}{2}} + \int\limits_{C_j} \mathcal{A}(\boldsymbol{P}_j(x))\,\frac{\mathrm{d}\boldsymbol{P}_j(x)}{\mathrm{d}x}\,\mathrm{d}x\Big)$$

where

$$\boldsymbol{D}^\pm_{j+\frac{1}{2}} = \frac{1 \pm \alpha_1^{j+\frac{1}{2}}}{2}\int\limits_0^1 \mathcal{A}(\boldsymbol{\Psi}_{j+\frac{1}{2}}(s))\,\frac{\mathrm{d}\boldsymbol{\Psi}_{j+\frac{1}{2}}}{\mathrm{d}s}\,\mathrm{d}s \pm \frac{\alpha_0^{j+\frac{1}{2}}}{2}\Big(\boldsymbol{U}^+_{j+\frac{1}{2}} - \boldsymbol{U}^-_{j+\frac{1}{2}}\Big)$$

Substituting $\mathcal{A}(\boldsymbol{U}) = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}(\boldsymbol{U}) - B(\boldsymbol{U})$ results in:

$$\int_{C_j} \mathcal{A}(\boldsymbol{P}_j(x)) \frac{\mathrm{d}\boldsymbol{P}_j(x)}{\mathrm{d}x} \,\mathrm{d}x = \int_{C_j} \left[\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}(\boldsymbol{P}_j(x)) - B(\boldsymbol{P}_j(x))\right] \frac{\mathrm{d}\boldsymbol{P}_j(x)}{\mathrm{d}x} \,\mathrm{d}x$$

$$= \boldsymbol{F}(\boldsymbol{U}^{-}_{j+\frac{1}{2}}) - \boldsymbol{F}(\boldsymbol{U}^{+}_{j-\frac{1}{2}}) - \int_{C_j} B(\boldsymbol{P}_j(x)) \frac{\mathrm{d}\boldsymbol{P}_j(x)}{\mathrm{d}x} \,\mathrm{d}x$$

$$\int_0^1 \mathcal{A}(\boldsymbol{\Psi}_{j+\frac{1}{2}}(s)) \frac{\mathrm{d}\boldsymbol{\Psi}_{j+\frac{1}{2}}}{\mathrm{d}s} \,\mathrm{d}s = \int_0^1 \left[\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}(\boldsymbol{\Psi}_{j+\frac{1}{2}}(s)) - B(\boldsymbol{\Psi}_{j+\frac{1}{2}}(s))\right] \frac{\mathrm{d}\boldsymbol{\Psi}_{j+\frac{1}{2}}}{\mathrm{d}s} \,\mathrm{d}s$$

$$= \boldsymbol{F}(\boldsymbol{U}^{+}_{j+\frac{1}{2}}) - \boldsymbol{F}(\boldsymbol{U}^{-}_{j+\frac{1}{2}}) - \int_0^1 B(\boldsymbol{\Psi}_{j+\frac{1}{2}}(s)) \frac{\mathrm{d}\boldsymbol{\Psi}_{j+\frac{1}{2}}}{\mathrm{d}s} \,\mathrm{d}s$$

Therefore, the semi-discrete central-upwind scheme reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\boldsymbol{U}}_j = -\frac{1}{\Delta x}\left(\boldsymbol{D}^+_{j-\frac{1}{2}} + \boldsymbol{D}^-_{j+\frac{1}{2}} + \boldsymbol{F}(\boldsymbol{U}^-_{j+\frac{1}{2}}) - \boldsymbol{F}(\boldsymbol{U}^+_{j-\frac{1}{2}}) - \boldsymbol{B}_j\right)$$

where

$$\boldsymbol{D}^{\pm}_{j+\frac{1}{2}} = \frac{1 \pm \alpha_1^{j+\frac{1}{2}}}{2}\Big(\boldsymbol{F}(\boldsymbol{U}^+_{j+\frac{1}{2}}) - \boldsymbol{F}(\boldsymbol{U}^-_{j+\frac{1}{2}}) - \boldsymbol{B}_{\boldsymbol{\Psi},j+\frac{1}{2}}\Big) \pm \frac{\alpha_0^{j+\frac{1}{2}}}{2}\Big(\boldsymbol{U}^+_{j+\frac{1}{2}} - \boldsymbol{U}^-_{j+\frac{1}{2}}\Big)$$

$$\boldsymbol{B}_j := \int\limits_{C_j} B(\boldsymbol{P}_j(x))\,\frac{\mathrm{d}\boldsymbol{P}_j(x)}{\mathrm{d}x}\,\mathrm{d}x, \qquad \boldsymbol{B}_{\boldsymbol{\Psi},j+\frac{1}{2}} := \int\limits_0^1 B(\boldsymbol{\Psi}_{j+\frac{1}{2}}(s))\,\frac{\mathrm{d}\boldsymbol{\Psi}_{j+\frac{1}{2}}}{\mathrm{d}s}\,\mathrm{d}s$$

We now can switch back to the flux form

Path-Conservative Central-Upwind (PCCU) Scheme

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\boldsymbol{U}}_j=-\frac{1}{\Delta x}\Bigg[\boldsymbol{H}_{j+\frac12}-\boldsymbol{H}_{j-\frac12}-\boldsymbol{B}_j-\frac{a^+_{j-\frac12}}{a^+_{j-\frac12}-a^-_{j-\frac12}}\boldsymbol{B}_{\boldsymbol{\Psi},j-\frac12}+\frac{a^-_{j+\frac12}}{a^+_{j+\frac12}-a^-_{j+\frac12}}\boldsymbol{B}_{\boldsymbol{\Psi},j+\frac12}\Bigg]$$

where

$$\boldsymbol{B}_j=\int\limits_{C_j}B(\boldsymbol{P}_j(x))\,\frac{\mathrm{d}\boldsymbol{P}_j(x)}{\mathrm{d}x}\,\mathrm{d}x,\qquad \boldsymbol{B}_{\boldsymbol{\Psi},j+\frac12}=\int\limits_0^1 B(\boldsymbol{\Psi}_{j+\frac12}(s))\,\frac{\mathrm{d}\boldsymbol{\Psi}_{j+\frac12}}{\mathrm{d}s}\,\mathrm{d}s$$

Remark: Treating the nonconservative product term $B(\boldsymbol{U})\boldsymbol{U}_x$ as a source term results in

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\boldsymbol{U}}_j=-\frac{1}{\Delta x}\Big[\boldsymbol{H}_{j+\frac12}-\boldsymbol{H}_{j-\frac12}-\boldsymbol{B}_j\Big]$$

Application to the Saint-Venant System

Example — Dam-Break Problem

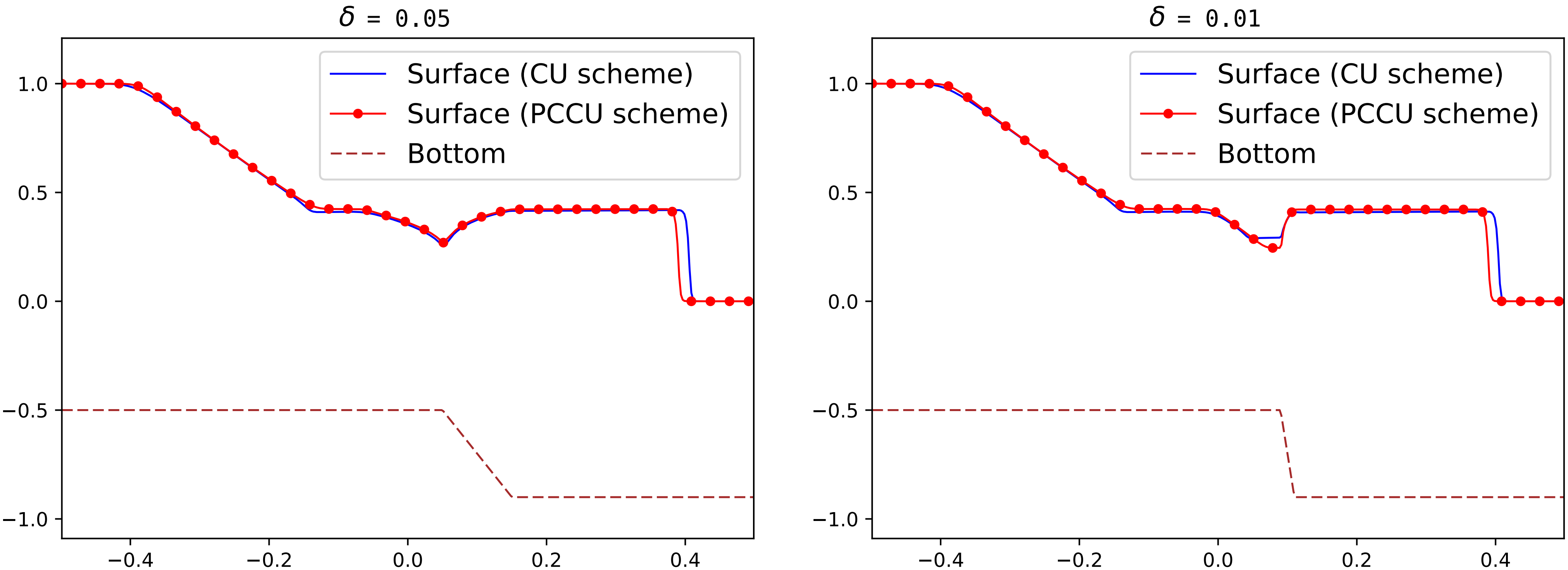
$$\omega(x,0) = h(x,0) + Z(x) = \begin{cases} 1, & \text{if } x < 0, \\ 0, & \text{if } x > 0, \end{cases} \qquad q(x,0) \equiv 0$$

$$Z(x) = \begin{cases} -0.5, & \text{if } x < 0.1 - \delta \\ -0.5 - \dfrac{0.2}{\delta}(x - 0.1 + \delta), & \text{if } 0.1 - \delta \le x \le 0.1 + \delta \\ -0.9, & \text{if } x > 0.1 + \delta \end{cases}$$

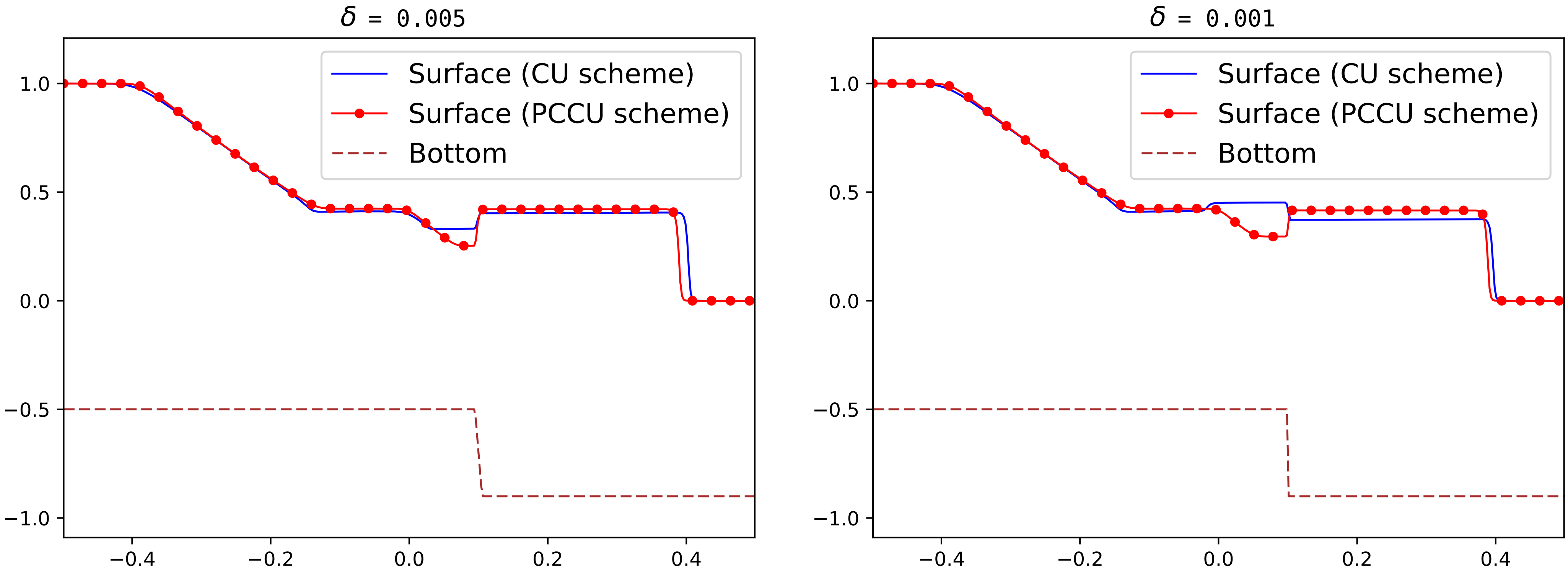
$\delta = 0.05$, 0.01, 0.005 and 0.001: parameter that is used to control the steepness of the slope in Z

Final time $t = 0.1$

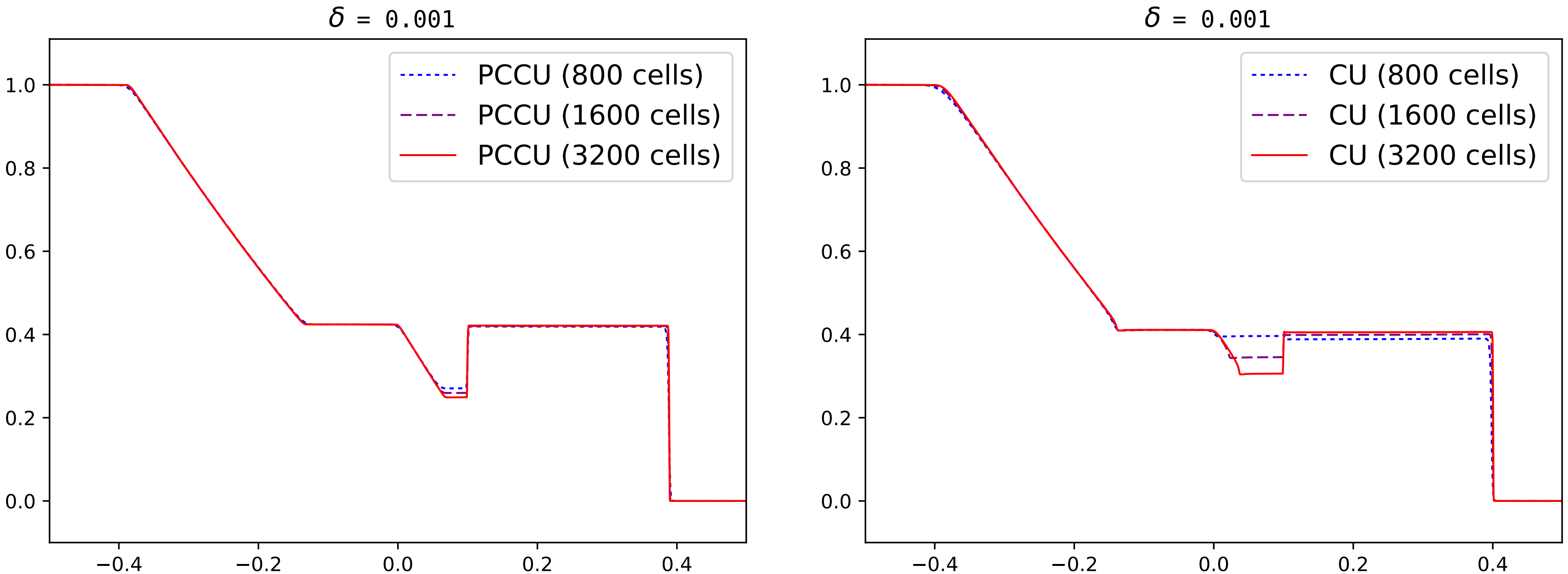
400 uniform finite-volume cells on the computational domain $[-0.5, 0.5]$



400 uniform finite-volume cells on the computational domain $[-0.5, 0.5]$



Experimental convergence



Application to the Two-Layer Shallow Water Equations

$$\begin{cases} (h_1)_t + (q_1)_x = 0 \\ (q_1)_t + \left(h_1 u_1^2 + \dfrac{g}{2} h_1^2\right)_x = -g h_1 (h_2 + Z)_x \\ (h_2)_t + (q_2)_x = 0 \\ (q_2)_t + \left(h_2 u_2^2 + \dfrac{g}{2} h_2^2\right)_x = -g h_2 (r h_1 + Z)_x \end{cases}$$

$h_1(x,t)$ and $h_2(x,t)$: depths of the upper (lighter) and lower (heavier) water layers

$q_1(x,t)$ and $q_2(x,t)$: their discharges

$u_1(x,t) := \dfrac{q_1(x,t)}{h_1(x,t)}$ and $u_2(x,t) := \dfrac{q_2(x,t)}{h_2(x,t)}$: their velocities

ρ_1 and ρ_2: their constant densities

$r := \rho_1/\rho_2 \le 1$: the density ratio

g: constant gravitational acceleration

$Z(x)$: bottom topography

Example — Riemann Problem with Initially Flat Water Surface

$$h_1(x,0) = \begin{cases} 1.8, & x < 0, \\ 0.2, & x > 0, \end{cases} \quad h_2(x,0) = \begin{cases} 0.2, & x < 0, \\ 1.8, & x > 0, \end{cases} \quad q_1(x,0) \equiv q_2(x,0) \equiv 0$$

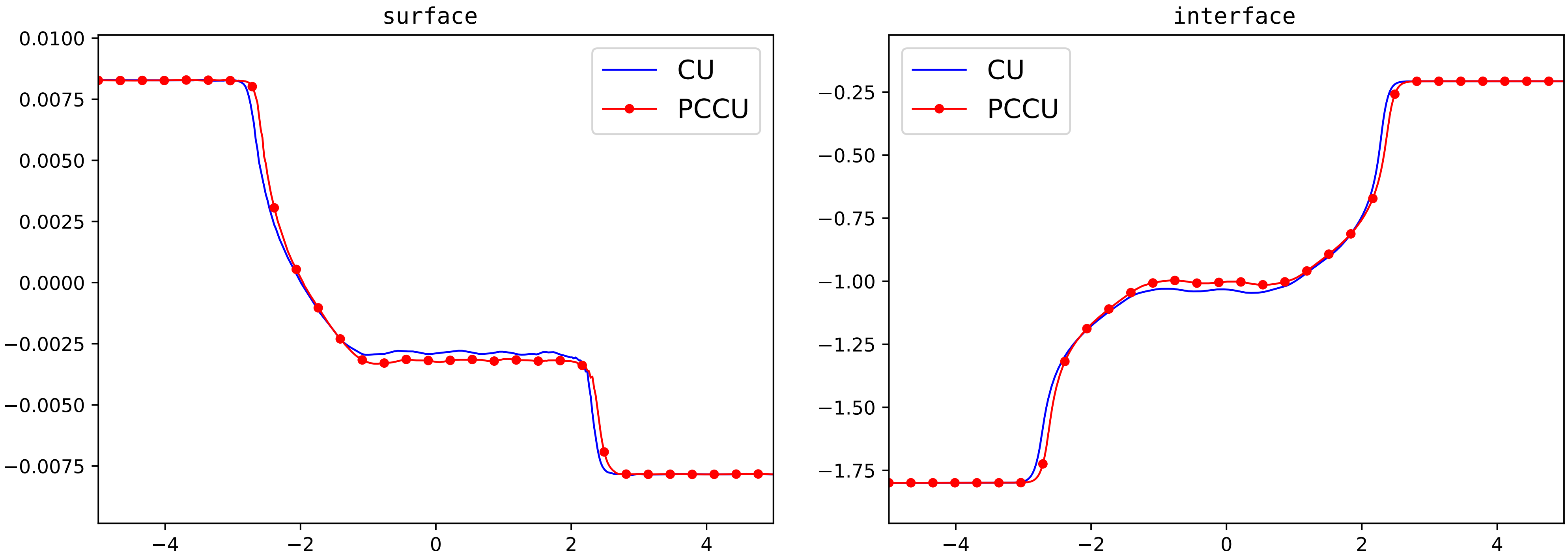
$$Z(x) \equiv -2$$

Note that initially the water surface is flat:

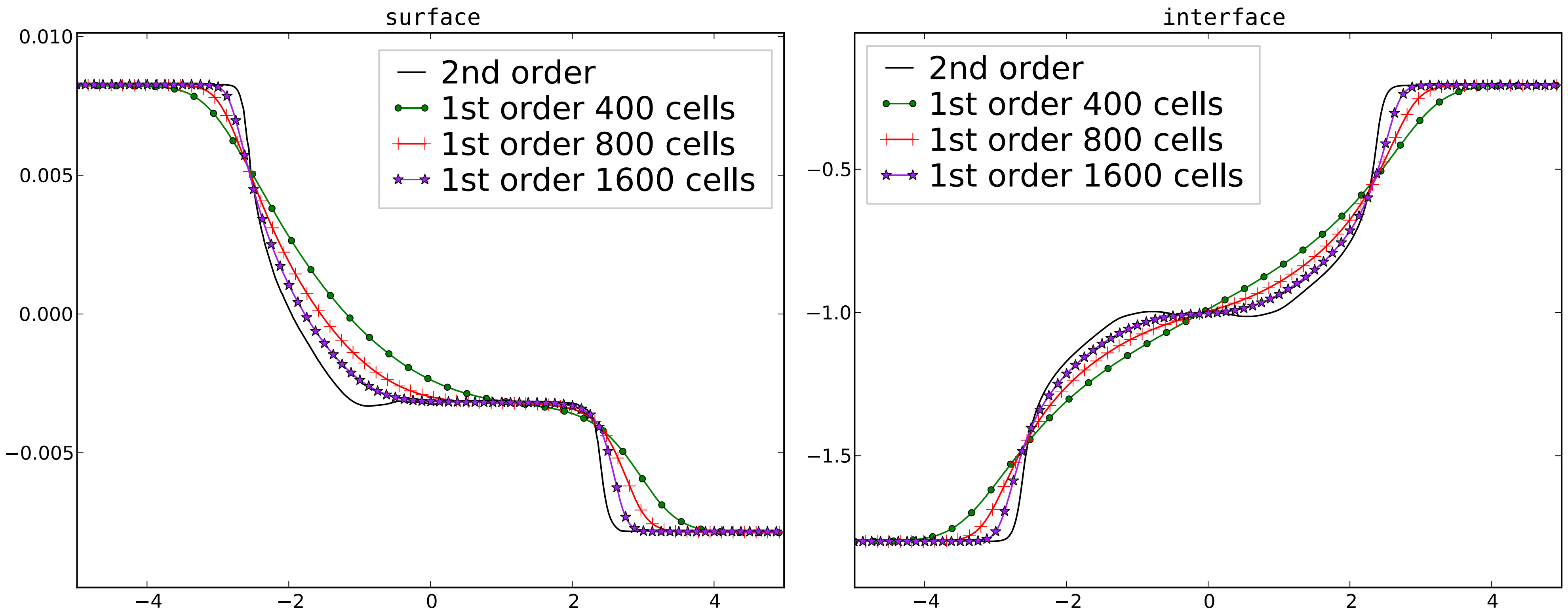
$$h_1(x,0) + h_2(x,0) + Z(x) \equiv 0$$

Final time $t = 7$

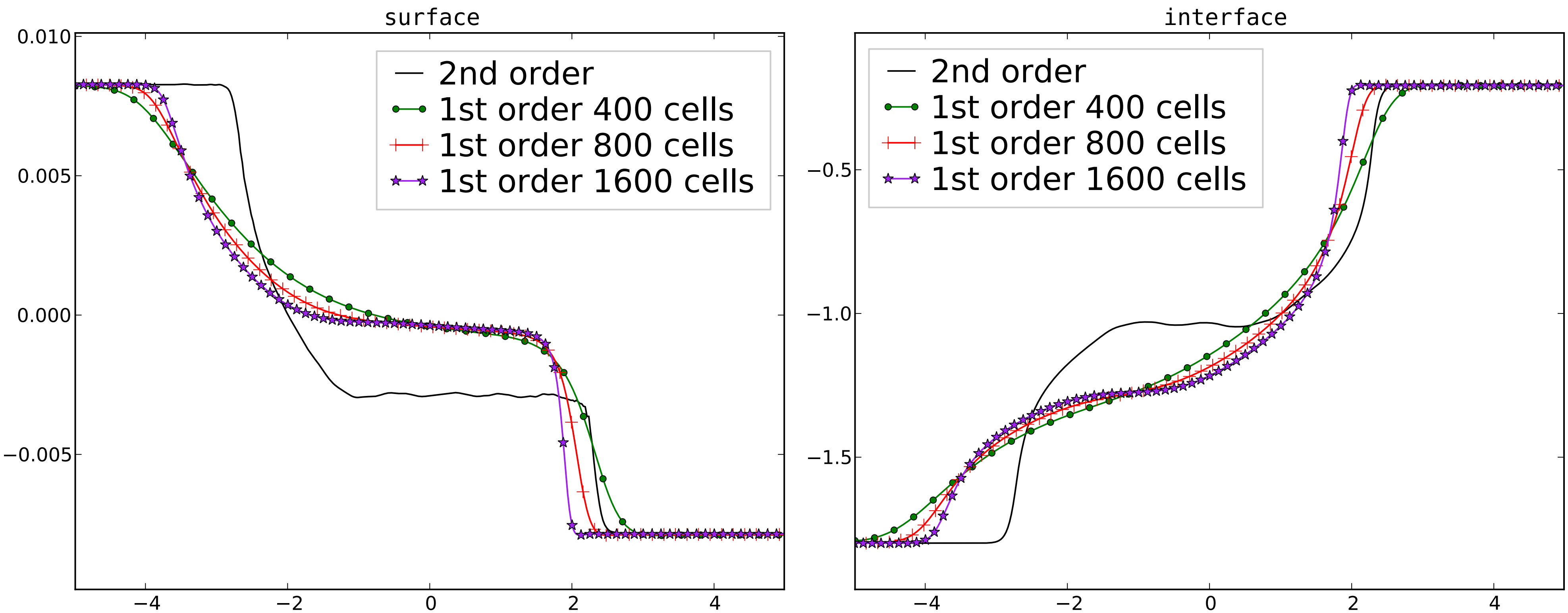
400 uniform finite-volume cells on the computational domain $[-0.5, 0.5]$



PCCU scheme: Experimental convergence of the first-order scheme



Original CU scheme: Experimental convergence of the first-order scheme



Example — Barotropic Tidal Flow

This example is designed to mimic a tidal wave by imposing periodic in time boundary conditions at the left end of the computational domain.

The initial data

$$\boldsymbol{U}(x,0) = \begin{cases} \boldsymbol{U}_L, & \text{if } x < 0 \\ \boldsymbol{U}_R, & \text{if } x > 0 \end{cases}$$

$$\boldsymbol{U}_R = \Big((h_1)_R, (q_1)_R, (h_2)_R, (q_2)_R\Big)^\top = \Big(0.37002, -0.18684, 1.5931, 0.17416\Big)^\top$$

$$\boldsymbol{U}_L = \Big((h_1)_L, (q_1)_L, (h_2)_L, (q_2)_L\Big)^\top = \Big(0.69914, -0.21977, 1.26932, 0.20656\Big)^\top$$

correspond to an isolated internal shock.

The bottom topography is flat:

$$Z(x) \equiv Z_{\text{ref}} = -\frac{1}{2}\Big((h_1)_L + (h_2)_L + (h_1)_R + (h_2)_R\Big)$$

We use 1000 uniform finite-volume cells on the computational domain $[-10, 10]$

We impose open boundary conditions on the right and the following periodic in time boundary conditions on the left for h_1 and h_2:

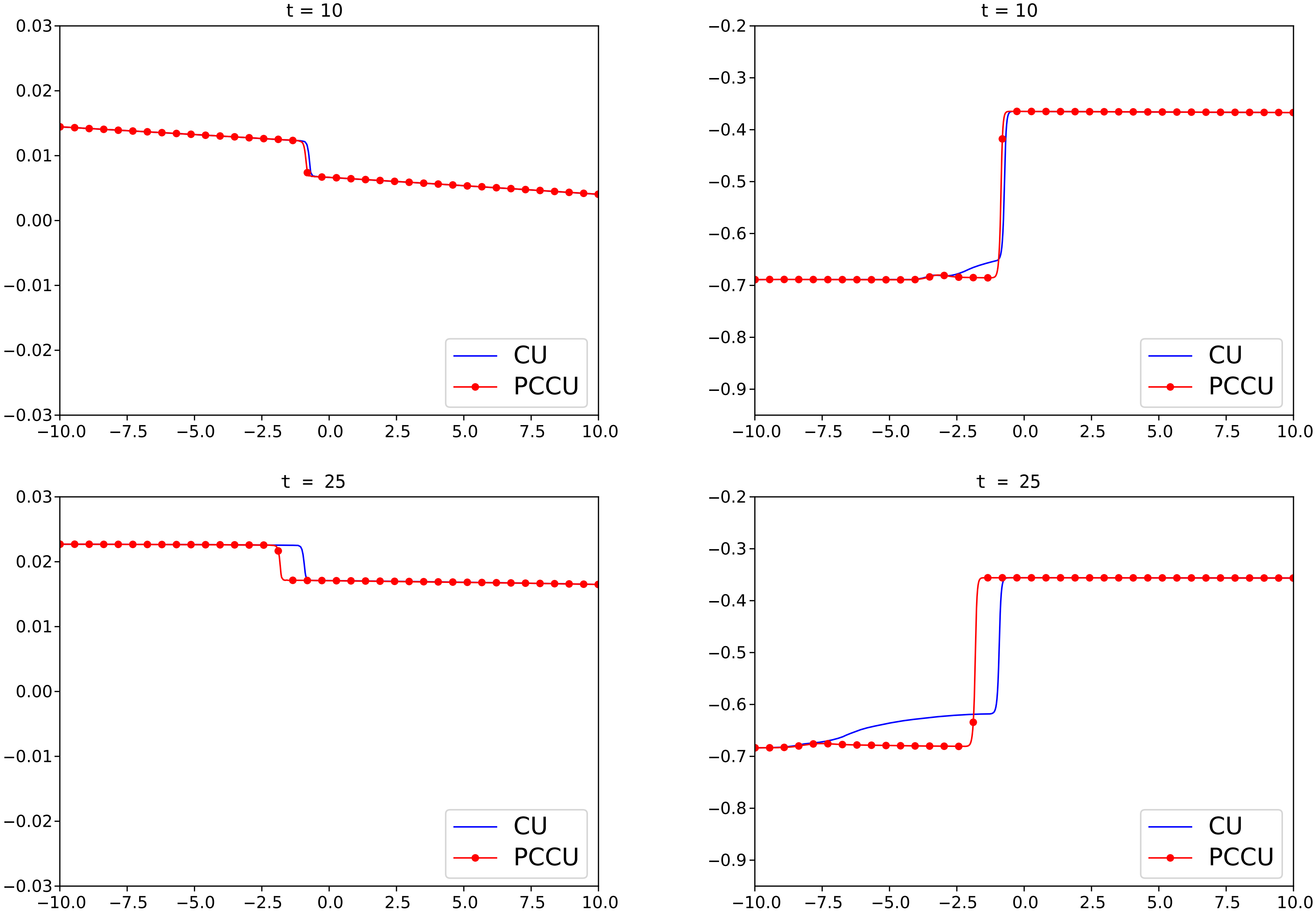
$$h_1(-10,t) = (h_1)_L + (h_1)_L \frac{0.03}{|Z_{\text{ref}}|} \sin\left(\frac{\pi t}{50}\right)$$

$$h_2(-10,t) = (h_2)_L + (h_1)_L \frac{0.03}{|Z_{\text{ref}}|} \sin\left(\frac{\pi t}{50}\right)$$

The values of q_1 and q_2 on the left are obtained by zero-order interpolation

$t \in [0, 64]$

Water surface $h_1 + h_2 + Z$ (left) and interface $h_2 + Z$ (right)



Water surface $h_1 + h_2 + Z$ (left) and interface $h_2 + Z$ (right)